Lecture 34

Summary so far. – Input: $A = (a_{ij})_{i,j\in I}$ Cartan matrix; $D = \mathrm{diag}(d_i : i\in I)$ symmetrizing integers.

$\rightsquigarrow$ $\mathfrak{g}(A)$ or just $\mathfrak{g}$. – finite-dim'l simple Lie alg. / $\mathbb{C}$ – with following present.

Generators: $h_i, e_i, f_i \quad (i\in I)$

Rel^{n}s: $[h_i, h_j] = 0 \quad (\forall\, i,j\in I)$

$$[h_i, e_j] = a_{ij}\, e_j \quad ; \quad [h_i, f_j] = -a_{ij}\, f_j$$

$$[e_i, f_j] = \delta_{ij}\, h_i$$

and for $i\neq j$; $\quad \mathrm{ad}(e_i)^{1-a_{ij}}.\, e_j = 0 = \mathrm{ad}(f_i)^{1-a_{ij}} f_j$

Set $\mathfrak{h} = \bigoplus_{i\in I} \mathbb{C} h_i$ (Cartan subalg.); and $\alpha_j \in \mathfrak{h}^*$ defined by $\alpha_j(h_i) = a_{ij}$ $\forall i\in I$.

$(\cdot,\cdot)$ on $\mathfrak{h}^*$: $\quad (\alpha_i, \alpha_j) = d_i a_{ij}$

Simple reflections – and Weyl group. – $\quad s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i \quad \forall\, \gamma\in\mathfrak{h}^*$, $i\in I$.

$$W = \langle s_i \rangle_{i\in I} \subset GL(\mathfrak{h}^*).$$

$$R = \bigcup_{i\in I} W\cdot\alpha_i \subset \mathfrak{h}^*\setminus\{0\}. \qquad R = R_+ \sqcup R_- \text{ (sign-coherence)}$$

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad ; \quad \mathfrak{n}_\pm = \bigoplus_{\alpha\in R_+} \mathfrak{g}_{\pm\alpha} \quad \left(\mathfrak{g}_\gamma = \{x\in\mathfrak{g} \mid [h,x] = \gamma(h)x \ \forall h\in\mathfrak{h}\},\ \gamma\in\mathfrak{h}^* \right)$$

$$\dim(\mathfrak{g}_\alpha) = 1 \quad \forall\, \alpha\in R.$$

§1. Category $\mathcal{O}$ and highest weight repns –

A $\mathfrak{g}$-repn V is <u>in category $\mathcal{O}$</u> if (i) V is $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces:

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu] \quad , \quad V[\mu] := \{ v \in V : h \cdot v = \mu(h) v \ \forall h \in \mathfrak{h} \}$$

$\dim V[\mu] < \infty$. Let $P(V) = \{ \mu \in \mathfrak{h}^* : V[\mu] \neq \{0\} \}$.

set of weights of V.

(ii) $\exists\ \lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ s.t. $P(V) \subset \bigcup_{j=1}^{r} \lambda_j - Q_+$.

(recall: $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathfrak{h}^*$)

A $\mathfrak{g}$-repn. V is said to be a <u>highest-weight repn. of highest weight</u> <u>$\Lambda \in \mathfrak{h}^*$</u> if

(i) $P(V) \subset \Lambda - Q_+$

(ii) $\dim V_{\Lambda}[\Lambda] = 1$, and

(iii) V is generated, as a $\mathfrak{g}$-repn., by $V_{\Lambda}[\Lambda]$. i.e.,

let $0 \neq v \in V_{\Lambda}[\Lambda]$. Then V is spanned by $\left\{ f_{i_1} \cdots f_{i_r} \cdot v \,\middle|\, \begin{matrix} r \geq 0 \\ i_1, \ldots, i_r \in I \end{matrix} \right\}$.

<u>Cor. of the existence of Casimir Operator.</u> -

Let V be a h.w. repn. of h.w. Λ. Assume $\exists\ 0 \neq u \in V[\lambda]$ s.t. $\mathfrak{n}_+ \cdot u = 0$. Then $|\lambda + \rho|^2 = |\Lambda + \rho|^2$

(recall: $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$; $\rho(h_i) = 1\ \forall i \in I$)

i.e. $\lambda \in \underbrace{B(\Lambda) := \{ \mu \in \Lambda - Q_+ : |\Lambda + \rho|^2 = |\mu + \rho|^2 \}}_{\text{a finite set.}}$

(Note $C\big|_V = (\Lambda + 2\rho, \Lambda) \cdot Id_V = \left(|\Lambda + \rho|^2 - |\rho|^2 \right) \cdot Id_V$

$C \cdot u = (\mu + 2\rho, \mu) \cdot u = \left(|\mu + \rho|^2 - |\rho|^2 \right) \cdot u$ $\square$)

Recall: $\forall\ \lambda \in \mathfrak{h}^*$, $\exists!$ h.w., irred., repn. of h.w. λ, denoted by L_λ. L_λ was constructed as the quotient of the Verma module M_λ by its unique, max'l, proper submodule $\mathcal{J}_\lambda \subset M_\lambda$.

§2. Characters. - Character of a f.d $\mathfrak{g}$-repn (resp. category $\mathcal{O}$ repn.) lies in the following polynomial ring $\mathbb{E}$ (resp. "power series" ring $\hat{\mathbb{E}}$).

Definition. - $\mathbb{E} := \left\{ \sum_{\mu \in \mathfrak{h}^*} c_\mu \cdot e^\mu \ : \ \begin{array}{l} c_\mu \in \mathbb{Z} \text{ and} \\ c_\mu = 0 \text{ for all but finitely} \\ \text{many } \mu\text{'s} \end{array} \right\}$

Mult. on $\mathbb{E}$ is given by $e^0 = 1$ and $e^\mu \cdot e^{\mu'} = e^{\mu+\mu'}$.

$\hat{\mathbb{E}} := \left\{ \sum_{\mu \in \mathfrak{h}^*} c_\mu \cdot e^\mu \ : \ \begin{array}{l} c_\mu \in \mathbb{Z} \text{ and } \exists\ \lambda_1, \dots, \lambda_r \in \mathfrak{h}^* \text{ s.t.} \\ \{\mu : c_\mu \neq 0\} \subset \bigcup_{j=1}^r \lambda_j - Q_+ \end{array} \right\}$

For $V \in \mathcal{O}$, define

$$\boxed{Ch(V) = \sum_{\mu \in \mathfrak{h}^*} (\dim V[\mu]) \cdot e^\mu} \qquad (Ch(V) \in \hat{\mathbb{E}})$$

Lemma. - (a) If $0 \to V' \to V \to V'' \to 0$ is a s.e.s. in category $\mathcal{O}$, then

$$Ch(V) = Ch(V') + Ch(V'').$$

(b) $\quad Ch(V_1 \otimes V_2) = Ch(V_1) \cdot Ch(V_2)$

(Proof is clear.)

§3. Characters of Verma Modules

Lemma. – $\quad Ch(M_\lambda) = \dfrac{e^{\lambda}}{\prod_{\alpha \in R_+} 1 - e^{-\alpha}} \qquad \forall \lambda \in \mathfrak{h}^*.$

Proof. – Recall $\dim M_\lambda[\lambda-\beta] = P_K(\beta) = \#$ of ways of writing β as a sum of positive roots.

$(\beta \in Q_+)$

$$M_\lambda = \bigoplus_{\beta \in Q_+} M_\lambda[\lambda-\beta]$$

$\left(\begin{array}{l} P_K(\beta) = \left| \left\{ \underline{a} = (a_\alpha) \in \mathbb{Z}_{\geq 0}^{R_+} : \sum_{\alpha \in R_+} a_\alpha \cdot \alpha = \beta \right\} \right| \\ \text{Kostant's partition fn.} \end{array} \right)$

$$\Rightarrow \quad Ch(M_\lambda) = \sum_{\beta \in Q_+} P_K(\beta) \cdot e^{\lambda-\beta}$$

$$= e^{\lambda} \cdot \left(\prod_{\alpha \in R_+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots) \right)$$

$$= e^{\lambda} \cdot \prod_{\alpha \in R_+} (1 - e^{-\alpha})^{-1} \qquad \square$$

Let $\Delta = \prod_{\alpha \in R_+} e^{\alpha/2} - e^{-\alpha/2}$. Then $Ch(M_\lambda) = \dfrac{e^{\lambda+\rho}}{\Delta}$.

$$= e^{\rho} \cdot \prod_{\alpha \in R_+} (1 - e^{-\alpha})$$

Note : $\forall\ i \in I$; $\boxed{s_i \cdot \Delta = -\Delta}$ because,

s_i preserves $R_+ \setminus \{\alpha_i\}$ and $s_i(\alpha_i) = -\alpha_i$. Thus,

$\Delta \in \mathbb{E}$ is W-<u>skew-invariant</u> ; i.e. $w \cdot \Delta = (-1)^{\ell(w)} \cdot \Delta$ $\forall\ w \in W$.

(here $W \circlearrowright \mathbb{E}$ as $w \cdot (e^{\mu}) = e^{w(\mu)}$ $\forall\ \mu \in \mathfrak{h}^*$).

§4. Weyl* Character formula .-

<u>Theorem.-</u> Let $\lambda \in P_+$ (i.e. $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i \in I$). Then,

$$\boxed{Ch(L_\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} 1 - e^{-\alpha}}}$$

Alternately written : $$ch(L_\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} \cdot e^{w(\lambda+\rho)}}{\Delta}$$

→ Compare with the expression for Schur poly.'s :

$$S_\lambda(x_1, \dots, x_N) = \frac{\sum_{\sigma \in S_N} \varepsilon(\sigma) \cdot \underline{x}^{\sigma(\lambda+\rho)}}{\prod_{i<j} x_i - x_j}$$ $\rho = (N-1, \dots, 0) \in \mathbb{Z}_{\geq 0}^N$

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* Hermann Klaus Hugo Weyl Nov. 9, 1885 - Dec. 8, 1955

The proof given here is due to Victor Kac. It rests on the following

Lemma. - Let $\Lambda \in \mathfrak{h}^*$. Define $B(\Lambda) := \{\lambda \in \Lambda - Q_+ : |\Lambda+\rho| = |\lambda+\rho|\}$

(a) For any highest weight repn V of $\mathfrak{g}$, of h.w. Λ, we have

$$\mathrm{Ch}(V) = \sum_{\mu \in B(\Lambda)} a_{V;\mu}\, \mathrm{Ch}(L_\mu), \quad \text{where } a_{V;\mu} \in \mathbb{Z}_{\geq 0} \text{ and } a_{V;\Lambda} = 1.$$

(b) $$\mathrm{Ch}(L_\Lambda) = \sum_{\mu \in B(\Lambda)} c_{\Lambda;\mu}\, \mathrm{Ch}(M_\mu), \quad \text{where } c_{\Lambda;\mu} \in \mathbb{Z} \text{ and } c_{\Lambda,\Lambda} = 1.$$

Proof. - (Uses Cor. of Casimir - see page 2 above)

(a) Let $N_V = \sum_{\lambda \in B(\Lambda)} \dim V[\lambda] < \infty$ as $|B(\Lambda)| < \infty$ and $\dim V[\mu] < \infty \ \forall \mu$.

Note - if V is irreducible, then $V \cong L_\Lambda$ and we are done.

Now we can argue by induction on N_V.

Assume V is not irred., then $\exists\ \lambda \in B(\Lambda)$, $\lambda < \Lambda$ s.t.

$$\{v \in V[\lambda] : \mathfrak{n}_+ v = 0\} \neq \{0\}.$$

Choose $0 \neq v \in V[\lambda]$ s.t. $\mathfrak{n}_+ v = 0$ and let $U \subset V$ be the subrepn generated by v.

$$0 \to U \to V \to V/U \to 0$$

By induction hypothesis $\mathrm{Ch}(U) = \sum_{\mu \in B(\lambda)} a_{U,\mu}\, \mathrm{ch}(L_\mu)$ and $\mathrm{ch}(V/U) = \sum_{\gamma \in B(\Lambda)} a_{V/U;\gamma}\, \mathrm{ch}(L_\gamma)$

as claimed in the assertion (a)

The proof follows from $Ch(V) = Ch(U) + ch(V/U)$

and $a_{U,\Lambda} = 0$; $a_{V/U:\Lambda} = 1$.

(b) Order the finite set $B(\Lambda) = \{\Lambda = \lambda_1, \lambda_2, \dots, \lambda_K\}$ so that $i > j$ implies $\lambda_i - \lambda_j \notin Q_+$. Using (a) for Verma Modules, we can write

$$Ch(M_{\lambda_i}) = \sum_{j=1}^{K} b_{ij}\, ch(L_{\lambda_j}) \qquad (b_{ij} \in \mathbb{Z}_{\geq 0}).$$

Note - if $i > j$, then $b_{ij} = 0$ since $\lambda_j \notin B(\lambda_i)$.

and $b_{ii} = 1$ by (a) above.

$\Rightarrow$ We can invert $(b_{ij})_{1 \leq i,j \leq K}$ to get

$$Ch(L_{\lambda_j}) = \sum_i c_{ij}\, ch(M_{\lambda_i})$$

Take $j = 1$, to get the assertion of Lemma (b). $\square$

§5. Proof of Weyl Character formula.

- $$Ch(L_\lambda) = \sum_{\mu \in B(\lambda)} c_\mu \cdot Ch(M_\mu)$$ $c_\mu \in \mathbb{Z}$; $c_\lambda = 1$. By Lemma (b) above

Multiply both sides by $\Delta = \prod_{\alpha \in R_+} e^{\alpha/2} - e^{-\alpha/2}$ to get

$$\Delta \cdot Ch(L_\lambda) = \sum_{\mu \in B(\lambda)} c_\mu \cdot e^{\mu + \rho}$$ (using Lemma §3 above)

Note : L_λ is f.d. $\Rightarrow$ $Ch(L_\lambda)$ is W-invariant

So, $\Delta \cdot Ch(L_\lambda)$ is W-skew-invariant. i.e.

$$\sum_{\mu \in B(\lambda)} c_\mu \cdot e^{\mu+\rho} = \sum_{\mu \in B(\lambda)} (-1)^{\ell(w)} c_\mu \cdot e^{w(\mu+\rho)} \quad \forall w \in W.$$

As $c_\lambda = 1$, we get

$$\Delta \cdot Ch(L_\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)} + \sum_{\substack{\mu \in B(\lambda) \\ \mu+\rho \notin W\cdot(\lambda+\rho)}} c_\mu \cdot e^{\mu+\rho}$$

It remains to show that the second term of the right-hand side above is zero.

i.e. <u>Claim</u>: $c_\mu \neq 0 \Rightarrow \mu+\rho \in W\cdot(\lambda+\rho)$.

<u>Proof</u>. Let $\mu_+ \in W\cdot(\mu+\rho) \cap P_+$ (Note - $\forall \gamma \in \mathfrak{h}_{\mathbb{R}}^*$, $\exists\, w \in W$ s.t. $w\gamma \in \mathfrak{h}^*_{,\mathbb{R};\geq 0} = \{\eta : \eta(h_i) \geq 0\}$).

By W-skew-symmetry, $c_{\mu_+-\rho} \neq 0$. Hence, $\mu_+ \leq \lambda+\rho$

$|\mu_+|^2 = |\lambda+\rho|^2$.

But this implies $\mu_+ = \lambda+\rho$ and we are done.

see Lecture 33 page 5

$(\lambda+\rho)(h_i) > 0 \ \forall i$
$\lambda+\rho-\mu_+ \in Q_+ \setminus \{0\}$ $\Rightarrow$ $(\lambda+\rho, \lambda+\rho-\mu_+) > 0$ i.e. $|\lambda+\rho|^2 > (\lambda+\rho, \mu_+)$.

$\mu_+(h_i) \geq 0 \ \forall i$ $\Rightarrow$ $(\mu_+, \lambda+\rho-\mu_+) \geq 0$. Hence $|\lambda+\rho|^2 > (\lambda+\rho, \mu_+) \geq |\mu_+|^2$.

$\square$