Lecture 35.

Recall - last time we proved the "Weyl character formula":

$$\boxed{\mathrm{Ch}(L_\lambda) = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+} 1-e^{-\alpha}}} \quad \forall \lambda \in P_+.$$

Here : (i) $\mathfrak{g}$ is a f.d. simple Lie alg. assoc. to a Cartan matrix $A=(a_{ij})_{i,j\in I}$

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

(ii) $\mathfrak{h} = \bigoplus_{i\in I} \mathbb{C} h_i \subset \mathfrak{g}$ is a Cartan subalgebra. The linear forms $\{\alpha_j \in \mathfrak{h}^*\}_{j\in I}$ are defined by $\alpha_j(h_i) = a_{ij}$ $\forall i \in I$.

(iii) $W \circlearrowright \mathfrak{h}^*$ by $s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i$ $\forall \gamma \in \mathfrak{h}^*, i\in I$.

$$R = \bigcup_{i\in I} W\alpha_i = R_+ \sqcup R_-.$$

Root and Weight lattices:

(iv) $Q = \sum_{i\in I} \mathbb{Z}\alpha_i \supset Q_+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$

$P = \{\gamma \in \mathfrak{h}^* : \gamma(h_i) \in \mathbb{Z}\ \forall i\} = \bigoplus_{i\in I} \mathbb{Z}\varpi_i$ ($\varpi_i \in \mathfrak{h}^*$ defined by $\varpi_i(h_j) = \delta_{ij}$)

$P_+ = \{\gamma \in P : \gamma(h_i) \geq 0, \forall i\}$

$$\rho = \frac{1}{2}\sum_{\alpha\in R_+} \alpha = \sum_{i\in I} \varpi_i \in P_+.$$

(v) $(\cdot,\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ given by $(\alpha_i,\alpha_j) = d_i a_{ij}$

($\{d_i\}_{i\in I}$ are symmetrizing integers).

Main steps in our proof.-

(a) Characters of Verma Modules :

$$\mathrm{Ch}(M_\mu) = e^{\mu} \cdot \prod_{\alpha \in R_+} (1 - e^{-\alpha})^{-1} = \frac{e^{\mu+\rho}}{\prod_{\alpha \in R_+} e^{\alpha/2} - e^{-\alpha/2}}$$

$$= e^{\mu} \cdot \sum_{\beta \in Q_+} \mathcal{P}_K(\beta) \cdot e^{-\beta} \quad \left(\mathcal{P}_K(\beta) = \text{Kostant partition fn.} \right)$$

(b) $\mathrm{Ch}(L_\lambda)$ is W- invariant - using "Weyl group action" on f.d. $\mathfrak{g}$- repns.

(c) $\mathrm{Ch}(L_\lambda)$ can be written as a $\mathbb{Z}$- linear combination of $\left\{ \mathrm{Ch}(M_\mu) : \mu \in B(\lambda) := \{\gamma \in \lambda - Q_+ : |\gamma+\rho| = |\lambda+\rho|\} \right\}$

$$\Rightarrow \quad \mathrm{Ch}(L_\lambda) = \sum_{\mu \in B(\lambda)} c_{\lambda\mu}\, \mathrm{Ch}(M_\mu) \; ; \quad c_{\lambda,\mu} \in \mathbb{Z} \text{ and } c_{\lambda\lambda} = 1.$$

[see Lemma §4 of Lecture 34 - page 6.]

(d) $\forall\, \mu \in P_+$, $\quad W\cdot\mu \cap P_+$ is a singleton.

$\lambda, \mu \in P_+ \; ; \; \mu \in B(\lambda) \quad \Rightarrow \quad \lambda = \mu.$

§1. Kostant's formula for dimensions of weight spaces.

Cor. - $$\boxed{\dim L_\lambda[\mu] = \sum_{w \in W} (-1)^{\ell(w)} \mathcal{P}_K\big(w(\lambda+\rho) - (\mu+\rho)\big)}$$

Convention : $\mathcal{P}_K(\gamma) = 0$ if $\gamma \notin Q_+$.

<u>Proof.</u> -

$$\mathrm{Ch}(L_\lambda) = \left(\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}\right) \cdot \prod_{\alpha\in R_+} (1-e^{-\alpha})^{-1}$$

$$= \left(\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}\right) \cdot \left(\sum_{\beta\in Q_+} \mathcal{P}_K(\beta)\, e^{-\beta}\right)$$

Coefficient of e^μ gives:

$$\dim L_\lambda[\mu] = \sum_{\substack{w\in W,\\ \beta\in Q_+ \text{ s.t.}\\ w(\lambda+\rho)-\rho-\beta=\mu}} (-1)^{\ell(w)} \mathcal{P}_K(\beta) = \sum_{w\in W} (-1)^{\ell(w)} \cdot \mathcal{P}_K(\beta)$$

$\beta = w(\lambda+\rho) - (\mu+\rho)$

and the corollary follows. □

§2. <u>Weyl dimension formula.</u> -

Theorem. -

$$\boxed{\dim L_\lambda = \prod_{\alpha\in R_+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}}$$

Compare with the following specialization of Schur functions:

[· $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N) \vdash n$

$$S_\lambda(\underbrace{1,\dots,1}_{\substack{N \text{ variables}\\ \text{set } =1}}) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j-i}$$

(also called Weyl dim. formula)]

<u>Proof.</u>- Using the character formula for $\lambda = 0$ so L_λ is 1-dim'l triv. repn:

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)-\rho} = \prod_{\alpha \in R_+} 1 - e^{-\alpha} \qquad - (1)$$

Now we "evaluate" both sides of the character formula at $t\rho$ and let $t \to 0$.

$$ev_\gamma(e^\alpha) := e^{(\alpha,\gamma)}$$

$$\lim_{t\to 0} ev_{t\rho}(Ch(L_\lambda)) = \sum_\mu (\dim L_\lambda[\mu])\, e^{t(\mu,\rho)} \Big|_{t=0} = \dim(L_\lambda). \qquad - (2)$$

$$ev_{t\rho}\left(\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}\right) = \sum_{w\in W} (-1)^{\ell(w)} e^{t(w(\lambda+\rho)-\rho,\rho)}$$

$$= e^{-t(\rho,\rho)} \cdot \sum_{w\in W} (-1)^{\ell(w)} e^{t(\lambda+\rho, w^{-1}\rho)} \qquad \begin{bmatrix} u = w^{-1} \\ \ell(u) = \ell(w) \end{bmatrix}$$

$$= e^{-t(\rho,\rho)} \cdot ev_{t(\lambda+\rho)}\left(\sum_{u\in W} (-1)^{\ell(u)} \cdot e^{u\rho}\right)$$

$$= e^{-t(\rho,\rho)} \cdot ev_{t(\lambda+\rho)}\left(e^{\rho} \cdot \prod_{\alpha\in R_+} 1 - e^{-\alpha}\right) \qquad \text{using (1) above}$$

$$= e^{t(\lambda,\rho)} \cdot \prod_{\alpha\in R_+} 1 - e^{-t(\lambda+\rho,\alpha)}$$

$$\Rightarrow\ ev_{t\rho}(\text{R.H.S. of Weyl char formula}) = e^{t(\lambda+\rho)} \cdot \prod_{\alpha\in R_+} \frac{1 - e^{-t(\lambda+\rho,\alpha)}}{1 - e^{-t(\rho,\alpha)}}$$

$$\lim_{t\to 0}\left(ev_{t\rho}\,(\text{R.H.S.})\right) = \prod_{\alpha\in R_+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)} \quad - \quad (3)$$

The dimension formula is obtained from (2) and (3) above. $\square$

§3. Examples. – (i) Type A_2. $(\alpha_1, \alpha_1) = 2 = (\alpha_2, \alpha_2)$
$(\alpha_1, \alpha_2) = -1.$

$(\omega_i, \alpha_j) = \delta_{ij}$. $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

$\rho = \alpha_1 + \alpha_2 = \omega_1 + \omega_2.$

$\alpha_1 = 2\omega_1 - \omega_2$
$\alpha_2 = -\omega_1 + 2\omega_2$

$$\dim L_{m\omega_1+n\omega_2} = \prod_{\alpha\in R_+} \frac{((m+1)\omega_1 + (n+1)\omega_2\,,\,\alpha)}{(\omega_1+\omega_2\,,\,\alpha)}$$

$$= \frac{(m+1)(n+1)(m+n+2)}{2}.$$

(ii) Type B_2 $A = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$ $d_1 = 1$; $d_2 = 2$

$R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$

$(\omega_1, \alpha_1) = 1$
$(\omega_2, \alpha_2) = 2\,\omega_2(h_2) = 2$ and $(\omega_i, \alpha_j) = 0$ if $i \neq j$

$\rho = \omega_1 + \omega_2.$

$$\Rightarrow \dim L_{m\omega_1+n\omega_2} = \frac{(m+1)(n+1)(m+n+2)(m+2n+3)}{6}$$

§4. Complete reducibility theorem. - Let $\lambda, \mu \in P_+$ and

$0 \to L_\lambda \to V \to L_\mu \to 0$ be a short exact seq. of $\mathfrak{g}$-repns.

Then $V \cong L_\lambda \oplus L_\mu$.

Note.- This implies that every f.d. $\mathfrak{g}$-repn. is completely reducible - i.e., iso. to a direct sum of irred. f.d. $\mathfrak{g}$-repns [see. Lecture 25, §4.]

Proof.- Case 1.- $\lambda = \mu$. In this case $V[\lambda]$ is 2-dim'l (note: f.d. $\Rightarrow$ $\mathfrak{h}$-diagonalizable)

So, if we choose a basis $\{v_1, v_2\}$ of $V[\lambda]$, then

V_i = subrepn. gen. by v_i $\cong L_\lambda$ $(i = 1, 2)$

and $V_1 \cap V_2$ is a proper subrepn of both V_1 & V_2 $\Rightarrow V_1 \cap V_2 = \{0\}$.

Hence $V \cong V_1 \oplus V_2 = L_\lambda \oplus L_\lambda$.

Case 2. - λ and μ are comparable - i.e. either $\lambda - \mu \in Q_+$ or $\mu - \lambda \in Q_+$. and $\lambda \neq \mu$.

If λ and μ are comparable and $\lambda, \mu \in P_+$, then

$$|\lambda + \rho| \neq |\mu + \rho|$$

$\Rightarrow$ Casimir acting on V has 2 distinct (generalized) eigenvalues

As Casimir operator commutes with $\mathfrak{g}$-action, $V \cong L_\lambda \oplus L_\mu$ as generalized eigenspaces of the Casimir

General case.- $0 \to L_\lambda \xrightarrow{i} V \xrightarrow{p} L_\mu \to 0$ s.e.s.

$\lambda, \mu \in P_+$;

λ and μ are incomparable

Choose $v_1, v_2 \in V$ s.t. $v_1 = i(\mathbb{1}_\lambda)$, $p(v_2) = \mathbb{1}_\mu$.

$V_1 =$ $\mathfrak{g}$-subrepn. generated by $v_1 \cong L_\lambda$. Moreover, $\forall\, i \in I$,

$e_i \cdot v_2 \in V_1[\mu + \alpha_i]$. So; if $e_i \cdot v_2 \neq 0$ for some $i \in I$, then $\mu + \alpha_i \leq \lambda$ contradicts the assumption that λ & μ are incomparable

Hence $V_2 =$ $\mathfrak{g}$-subrepn. gen. by $v_2 \cong L_\mu$ and

$V \cong V_1 \oplus V_2$. ($V_1 \cap V_2 = \{0\}$ by the same logic as in Case 1.)

$\square$