# Lecture 36

## Summary of the course -

| Algebraic object | Repns | Complete Reducibility | Irred. Repns. | Characters |
|---|---|---|---|---|
| $S_n$ | finite-dim'l over $\mathbb{C}$ | ✓ | Partitions $\lambda \vdash n$ | $\chi_\lambda = s_\lambda$ via Frobenius char. map. |
| $GL_m(\mathbb{C})$ | Poly. repns. over $\mathbb{C}$ | ✓ | $\bigcup_n \{\lambda \vdash n \mid \ell(\lambda) \leq m\}$ | $\chi_{L_\lambda}(x_1, \ldots, x_m) = s_\lambda(x_1, \ldots, x_m)$ |
| $\mathfrak{g}(A)$ | f.d./$\mathbb{C}$ | ✓ | $P_+$ | Weyl char formula |
| | $\mathcal{O}$ | × | $\mathfrak{h}^*$ | $Ch(M_\lambda) = \dfrac{e^\lambda}{\prod_{\alpha \in R_+} 1 - e^{-\alpha}}$ |

Further directions.

- $\mathfrak{g} \rightsquigarrow \hat{\mathfrak{g}}$ (affine Lie algebras)

- $\mathfrak{g} \rightsquigarrow \mathcal{U}_q(\mathfrak{g})$ (quantization)

## §1. Some historical notes.-

(a) The proof of Weyl character formula, presented in this course, is taken from Kac's book "Infinite-dimensional Lie algebras". Kac attributes the proof to the following foundational paper:

Bernstein-Gelfand-Gelfand, Structure of representations generated by highest weight vectors, Funct. Analysis & app. (1971).

(b) Considering the Weyl denominator identity (generalization of van der Monde determinant identity):

$$\boxed{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)-\rho} = \prod_{\alpha \in R_+} 1 - e^{-\alpha}}$$

in 1972, I.G. Macdonald (Affine root systems & Dedekind's $\eta$-fn. - Invent. Math. 1972) observed that this identity remains valid for infinite Weyl groups as well. Except - one needs to introduce some "mysterious term" to the R.H.S. (see §2 below). The appearance of these "mysterious" factors was explained by Kac (1974 - Infinite-dim'l Lie alg's & Dedekind $\eta$-fn. Funct. analysis & app.) who also generalized Weyl character formula to this new family of Lie algebras, now called Kac-Moody algebras.

(c) Kac's work on generalizing f.d. simple Lie alg's to Kac-Moody algebras was inspired by his earlier research on finite order automorphisms of simple Lie algebras. (1967 – 1969).

§2. Example of $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. Macdonald's "mysterious term".

We will follow the same computation .. $\alpha_0, \alpha_1 \in \mathfrak{h}^*$ $\quad \alpha_j(h_i) = a_{ij}$

$h_0, h_1 \in \mathfrak{h}$

Note: $\alpha_0 + \alpha_1 (h_j) = 0 \quad \forall j = 0, 1$. So, we may have to "enlarge" $\mathfrak{h}$ if we want $\{\alpha_0, \alpha_1\}$ and $\{h_0, h_1\}$ to be linearly independent sets. So, assume $\dim \mathfrak{h} \geq 3$.

$$s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i \quad (i = 0, 1) \quad . \quad W = \langle s_0, s_1 \rangle \quad . \quad R = \bigcup_{i=0,1} W \cdot \alpha_i$$

Simple calculation gives $R = \pm \{ (m+1)\alpha_0 + m\alpha_1 ,\ m\alpha_0 + (m+1)\alpha_1 : m \in \mathbb{Z}_{\geq 0} \}$

$$s_0 \cdot \big( (m+1)\alpha_0 + m\alpha_1 \big) = -(m+1)\alpha_0 + m(\alpha_1 + 2\alpha_0)$$
$$= (m-1)\alpha_0 + m\alpha_1$$

$$s_0 \cdot \big( m\alpha_0 + (m+1)\alpha_1 \big) = -m\alpha_0 + (m+1)(\alpha_1 + 2\alpha_0)$$
$$= (m+2)\alpha_0 + (m+1)\alpha_1$$

(Flip $0 \leftrightarrow 1$ to get $s_1$ action).

Candidate for Weyl denominator (RHS) $= \prod_{\alpha \in R_+} 1 - e^{\alpha} = \prod_{m \in \mathbb{Z}_{\geq 0}} (1 - e^{-(m+1)\alpha_0 - m\alpha_1})$.
$(1 - e^{-m\alpha_0 - (m+1)\alpha_1}$

$$= \prod_{m \geq 0} \left(1 - u^{m+1} v^{m}\right)\left(1 - u^{m} v^{m+1}\right) \quad - (1)$$

[In Macdonald's notation $u = e^{-\alpha_0}$ ; $v = e^{-\alpha_1}$]

For the L.H.S., choose $\rho \in \mathfrak{h}^*$ s.t. $\rho(h_i) = 1$ $(i = 0, 1)$

i.e., $s_i(\rho) = \rho - \alpha_i$ $(i = 0, 1)$. <u>Note</u>. $\dim \mathfrak{h} \geq 3$ - so $\rho$ is not uniquely determined - but $w(\rho) - \rho$ $(w \in W)$ is.

Second calculation.-

$$\underbrace{s_0 s_1 \cdots s_1 s_0}_{m \text{ odd terms}} \cdot \rho = \rho - \frac{m(m+1)}{2}\alpha_0 - \frac{m(m-1)}{2}\alpha_1$$

$$\underbrace{s_1 s_0 \cdots s_1 s_0}_{n : \text{ even terms}} \cdot \rho = \rho - \frac{(n-1)n}{2}\alpha_0 - \frac{n(n+1)}{2}\alpha_1$$

(Flip $0 \leftrightarrow 1$ to get $s_1 s_0 \cdots s_0 s_1 \cdot \rho$ and $s_0 s_1 \cdots s_0 s_1 \cdot \rho$)

L.H.S. of the denominator id.

$$= \sum_{\ell \in \mathbb{Z}} (-1)^{\ell} u^{\frac{\ell(\ell+1)}{2}} v^{\frac{\ell(\ell-1)}{2}} \quad - (2)$$

$\rightarrow$ (1) and (2) are not equal. In fact, the following identity is well-known (Jacobi's triple product id. ~1829):

$$\boxed{\sum_{\ell \in \mathbb{Z}} (-1)^{\ell} u^{\frac{\ell(\ell+1)}{2}} v^{\frac{\ell(\ell-1)}{2}} = \prod_{n \geq 1} \left(1 - u^{n} v^{n}\right)\left(1 - u^{n-1} v^{n}\right)\left(1 - u^{n} v^{n-1}\right)}$$

Remark.- Set $q = uv$ and $z = u$. Then, the Jacobi identity takes a more familiar form:

$$\sum_{\ell \in \mathbb{Z}} (-1)^\ell q^{\frac{\ell(\ell-1)}{2}} \cdot z^\ell = \left(\prod_{n=1}^{\infty} 1-q^n\right)\left(\prod_{n\geq 1}(1-q^n z^{-1})(1-q^{n-1}z)\right)$$

For $|q|<1$, both sides converge uniformly on compact subsets in $\mathbb{C}\setminus\{0\}$. This is (one of the four) theta function,

§3.- Affine Lie algebra $\widetilde{sl_2}$ is defined as a central extension of $sl_2[t,t^{-1}] = sl_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$; plus a derivation.

(called "loop algebra")

$$\widetilde{sl_2} = sl_2[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \quad \text{as vector space.}$$

Lie bracket: (notation: $x \in sl_2$; $n \in \mathbb{Z}$, $x(n) = x \otimes t^n \in sl_2[t,t^{-1}]$)

- $[x(n), y(m)] = [x,y](n+m) + \delta_{n+m,0} \cdot \underbrace{(x,y)} \cdot nc$

  [non-deg. inv. form: $(e,f)=1$, $(h,h)=2$, all other $0$

- $[c, -] \equiv 0$ ($c$ is central)
- $[d, x(n)] = n\,x(n)$

(so $\mathrm{ad}(d) = t \cdot \frac{d}{dt}$.)

Cartan subalgebra : $\tilde{\mathfrak{h}} = \mathbb{C}\cdot h \oplus \mathbb{C}c \oplus \mathbb{C}d$ (3-dim'l)

Roots as $\mathrm{ad}(\tilde{\mathfrak{h}})$ eigenvalues on $\tilde{\mathfrak{g}}$ $(\mathfrak{g} = \mathfrak{sl}_2)$

Basis of $\widetilde{\mathfrak{sl}_2}$ : $\{h(n), e(n), f(n) : n \in \mathbb{Z}\} \cup \{c, d\}$

$\mathbb{C}\cdot h(n) = \widetilde{\mathfrak{sl}_2}_{\,n\delta}$ where $\delta : \tilde{\mathfrak{h}} \to \mathbb{C}$ is given by
$(n \neq 0)$ $\delta(h) = 0$; $\delta(c) = 0$; $\delta(d) = 1$.

$\mathbb{C}\cdot e(n) = \widetilde{\mathfrak{sl}_2}_{\,\alpha + n\delta}$ where $\alpha \in \tilde{\mathfrak{h}}^*$ : $\alpha(h) = 2$
$(n \in \mathbb{Z})$ $\alpha(c) = 0$, $\alpha(d) = 0$.

$\mathbb{C}\cdot f(n) = \widetilde{\mathfrak{sl}_2}_{\,-\alpha + n\delta}$
$(n \in \mathbb{Z})$

So $\tilde{R} := \{\gamma \in \tilde{\mathfrak{h}}^* \setminus \{0\} : \widetilde{\mathfrak{sl}_2}_{\,\gamma} \neq \{0\}\} = \{\pm\alpha + n\delta : n \in \mathbb{Z}\} \cup \underbrace{\{n\delta : n \in \mathbb{Z}_{\neq 0}\}}_{\text{"mysterious term comes from these"}}$

A "positive system" :

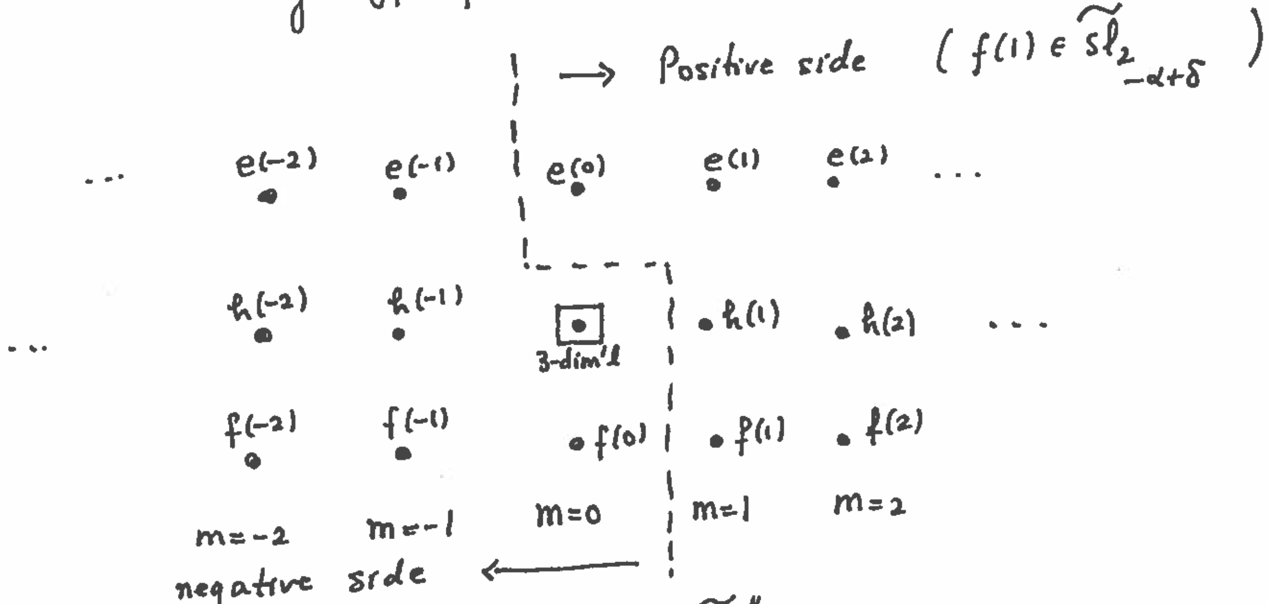
Set $\alpha_0 = -\alpha + \delta$.

Then $\tilde{R} = \tilde{R}_+ \sqcup (-\tilde{R}_+)$ where

$$\tilde{R}_+ = \tilde{R} \cap \left(\sum_{i=0,1} \mathbb{Z}_{\geq 0}\alpha_i\right) = \{n\delta : n \geq 1\} \cup \{\alpha + k\delta : k \geq 0\} \cup \{-\alpha + n\delta : n \geq 1\}$$

(compare with 3 indices in Jacobi's triple product id.)

## §4. Chevalley-type presentation of $\widetilde{sl_2}$.




" Picture of $\widetilde{sl_2}$ "

$\widetilde{sl_2}$ is generated by $\tilde{\mathfrak{h}}$ ; $\begin{matrix} f(0) \\ e(-1) \end{matrix}$ ; $\begin{matrix} e(0) \\ f(1) \end{matrix}$ .

↑ 3-dim'l

Notation change. $h_1 = h(0)$ ; $e_1 = e(0)$ ; $f_1 = f(0)$ (usual $sl_2 \subset sl_2[t,t^{-1}]$)

$$h_0 = c - h(0) \; ; \quad e_0 = f(1) \; ; \quad f_0 = e(-1)$$

[subscripts form the indexing set for $A = \begin{array}{cc} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \\ \begin{array}{cc} 0 & 1 \end{array} & \end{array}$ .]

Then ; (i) $\tilde{\mathfrak{h}} \subset \widetilde{sl_2}$ is abelian. $\left( ad(d) = \begin{matrix} 0 \text{ on } e_1, f_1 \\ 1 \text{ on } e_0 \; ; \; -1 \text{ on } f_0 \end{matrix} \right)$.

(ii) $[h_i, e_j] = a_{ij} e_j$ ; $[h_i, f_j] = -a_{ij} f_j$.

(iii) $[e_1, f_1] = h_1$ ; $[e_0, f_0] = h_0$.

$$[e_i, f_j] = \delta_{ij} h_i$$

(iv) $ad(e_0)^3 e_1 = ad(e_1)^3 e_0 = 0$ (same for $f$'s)