PROBLEMS IN REPRESENTATION THEORY

In Problems 1–10, k is a field and A is a unital, associative algebra over k.

Problem 1. [10 points] Let V be a representation of A.

(a) Prove that V is irreducible if, and only if, for every non–zero $v \in V$, we have: $A \cdot v = V$. Here, $A \cdot v := \{a \cdot v : a \in A\}$.

(b) We say that V is co–cyclic if there exists $0 \neq v \in V$ such that $v \in U$ for every non–zero subrepresentation $U \subset V$. Prove that every co–cyclic representation is indecomposable.

(c) Let $A = \mathbb{C}[x, y]/(x^2, xy, y^2)$. Prove that A is indecomposable (as A–representation) and A is not co–cyclic. Consider $\rho : A \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^3)$ given by:

$$\rho(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \rho(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that ρ a co–cyclic (hence indecomposable) representation of A.

Problem 2. [15 points] Let K be an extension of k (i.e., k is a subfield of K), and let $f_1, \ldots, f_m : A \to K$ be m distinct homomorphisms of k–algebras. Prove that $f_1, \ldots, f_m$ are linearly independent over K.

Problem 3. [10 points] Assume that k is algebraically closed in this problem. Let V be a finite–dimensional representation of A. Prove that V is semisimple if, and only if, for every subrepresentation $U \subset V$, there exists a subrepresentation $W \subset V$ such that $V \cong U \oplus W$.

Problem 4. [10 points] Show that the following representations are irreducible.

(a) $A = \mathrm{Mat}_{n\times n}(k)$, $V = k^n$.

(b) $A = \mathbb{C}\langle X, D\rangle/(DX - XD - 1)$, $V = \mathbb{C}[x]$ with the action given by $X \cdot f(x) = xf(x)$ and $D \cdot f(x) = f'(x)$ (derivative of f).

Problem 5. [10 points] (Idempotent decomposition) Assume that there exist $e_1, \ldots, e_n \in A$ such that

$$1 = e_1 + \cdots + e_n, \qquad e_i e_j = \delta_{ij} e_i, \ \forall\ 1 \leq i, j \leq n.$$

Prove that for any A–representation V, we have a direct sum decomposition of vector spaces: $V \cong \oplus_{i=1}^n e_i \cdot V$.

Problem 6. [10 points] (Tensor-Hom adjointness) Let V and W be two vector spaces over k. Write the natural map $\alpha_{V,W} : W \otimes V^* \to \mathrm{Hom}_k(V, W)$. Prove that this map is an isomorphism, assuming V is finite–dimensional. Here, $V^* = \mathrm{Hom}_k(V, k)$ is the dual vector space.

Problem 7. [10 points] Assume that k is algebraically closed. Let V be a finite–dimensional, irreducible representation of A. Let D_1 and D_2 be two finite–dimensional vector spaces over k. Recall that we have an isomorphism:

$$\mathrm{Hom}_k(D_1, D_2) \xrightarrow{\varphi} \mathrm{Hom}_A(V \otimes_k D_1, V \otimes_k D_2)$$

given by: $\varphi(X)(v \otimes \xi) = v \otimes X(\xi)$, for every $X \in \mathrm{Hom}_k(D_1, D_2)$, $v \in V$ and $\xi \in D_1$. Prove that (a) $\mathrm{Ker}(\varphi(X)) = V \otimes \mathrm{Ker}(X)$ and (b) $\mathrm{Image}(\varphi(X)) = V \otimes \mathrm{Image}(X)$.

Problem 8. [10 points] Let V, W be two finite–dimensional representations of A. Assume that there exists a finite extension L of k such that $V \otimes_k L$ is isomorphic to $W \otimes_k L$, as representations of $A \otimes_k L$. Prove that V and W are isomorphic (as representations of A). [Problem 3.8.4 of Pasha's book.]

Problem 9. [15 points] Let $A = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous } | f(x+1) = f(x)\}$ and let $M = \{g : \mathbb{R} \to \mathbb{R} \text{ continuous } | g(x+1) = -g(x)\}$. Prove that A and M are non–isomorphic, indecomposable representations of A. Show that $A \oplus A$ is isomorphic to $M \oplus M$.

Problem 10. [10 points] (See Problem 3.9.1 of Pasha's book). Let V, W be two representations of A. Given a linear map $f : A \to \mathrm{Hom}_k(W, V)$, define $\rho_f : A \to \mathrm{End}_k(V \oplus W)$ given by:

$$\rho_f(a) = \begin{bmatrix} \rho_1(a) & f(a) \\ 0 & \rho_2(a) \end{bmatrix}.$$

(a) Show that ρ_f is a representation of A if, and only if $f(ab) = \rho_1(a)f(b) + f(a)\rho_2(b)$, for every $a, b \in A$. Let the space of all the solutions of this equation be denoted by $Z^1(W, V) \subset \mathrm{Hom}_k(A, \mathrm{Hom}_k(W, V))$ (called 1–cocycles).

(b) Given $X \in \mathrm{Hom}_k(W, V)$, define the automorphism of the vector space $g \in \mathrm{Aut}_k(V \oplus W)$ given by:

$$g = \begin{bmatrix} \mathrm{Id}_{V_1} & X \\ 0 & \mathrm{Id}_{V_2} \end{bmatrix}.$$

Let $f : A \to \mathrm{Hom}_k(W, V)$ be an element of $Z^1(W, V)$ as in part (a). Show that g is an isomorphism between ρ_f and ρ_0 if, and only if $f(a) = \rho_1(a)X - X\rho_2(a)$ for every $a \in A$. Given $X \in \mathrm{Hom}_k(W, V)$, let $\delta(X) : A \to \mathrm{Hom}_k(W, V)$ be defined as: $\delta(X)(a) = \rho_1(a)X - X\rho_2(a)$.

Problem 10 cntd.. [20 points] (same notations as in Problem 10 above). Let $B^1(W, V) = \{\delta(X) : X \in \mathrm{Hom}_k(W, V)\}$. Verify that $B^1(W, V) \subset Z^1(W, V)$. Define $\mathrm{Ext}^1_A(W, V) := Z^1(W, V)/B^1(W, V)$.

Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of A–linear maps between A–representations. Show that, for every representation W of A, we get the following long exact sequence of vector spaces (describe the connecting homomorphism δ explicitly):

$$0 \longrightarrow \mathrm{Hom}_A(W, V_1) \longrightarrow \mathrm{Hom}_A(W, V_2) \longrightarrow \mathrm{Hom}_A(W, V_3) \xrightarrow{\delta} \mathrm{Ext}^1(W, V_1) \longrightarrow \mathrm{Ext}^1(W, V_2) \longrightarrow \mathrm{Ext}^1(W, V_3)$$

Problem 11. **[15 points]** Let G be a finite group acting on a finite set X via group homomorphism $\alpha : G \to S_X$ (here, S_X is the group of permutations of elements of X). Let V be the vector space (over $\mathbb{C}$) of all complex–valued functions on X. We view V as a G–representation, $\rho : G \to \mathrm{GL}(V)$, by:

$$(\rho(g)(f))(x) := f(\alpha(g)^{-1}(x)), \text{ for every } g \in G, f \in V, x \in X.$$

(1) Prove that $\dim(V^G) = |X/G|$ (number of G–orbits in X).
(2) For $g \in G$, show that $\mathrm{Tr}(\rho(g)) = |X^g|$. Here, $X^g := \{x \in X : \alpha(g)(x) = x\}$.
(3) Consider the averaging operator $P = \frac{1}{|G|}\sum_{g\in G} \rho(g)$, acting on V. Show that $\mathrm{Tr}(P) = \dim(V^G)$. Hence we obtain *Burnside's orbit counting formula*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

Problem 12. **[10 points]** Let G be a finite group and V a finite–dimensional G–representation over $\mathbb{C}$. Assume that $\chi_V(g) = 0$ for every $g \in G \setminus \{e\}$. Prove that $\dim(V)$ is divisible by $|G|$.

Problem 13. **[20 points]** Let D_n be the dihedral group (symmetries of regular n–gon, thus $|D_n| = 2n$). In the problems below, the representations of D_n are over $\mathbb{C}$.

(1) Count the number of conjugacy classes in D_n.
(2) Show that D_n has exactly two 1–dimensional representations, if n is odd; and exactly four 1–dimensional representations, if n is even.
(3) Let V be an irreducible representation of D_n. Show that $\dim(V) \leq 2$.
(4) Give explicit description of all irreducible D_n–representations.

Problem 14. **[10 points]** Let S_n be the group of permutations of n letters. Let $\mathbb{C}^n$ be the standard n–dimensional S_n–representation (i.e., in a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$: $\sigma \cdot e_i = e_{\sigma(i)}$, for every $\sigma \in S_n$ and $1 \leq i \leq n$). Show that $\mathbb{C}^n$ is a direct sum of two irreducible S_n–representations. Give a basis of each of these subrepresentations of $\mathbb{C}^n$.

Notations for Problems 15 and 16: Let G be a finite group, $\mathbb{C}[G]$ its group algebra over $\mathbb{C}$. Recall the convolution product, for $f_1, f_2 \in \mathbb{C}[G]$:

$$(f_1 * f_2)(g) = \sum_{\sigma\in G} f_1(\sigma) f_2(\sigma^{-1}g).$$

We defined a symmetric, bilinear form $\beta : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$ by:

$$\beta(f_1, f_2) = \frac{1}{|G|}(f_1 * f_2)(e) = \frac{1}{|G|}\sum_{g\in G} f_1(g^{-1}) f_2(g) \ .$$

Let $\mathbb{C}[G]_{\text{class}}$ denote the (commutative) subalgebra of functions constant on conjugacy classes of G (abbreviated to $\mathrm{Conj}(()G)$).

Let $\{V_\lambda : \lambda \in P(G)\}$ be the set of isomorphism classes of finite–dimensional, irreducible G–representations. Let ρ_λ denote the group homomorphism $G \to \mathrm{GL}(V_\lambda)$, or the corresponding algebra homomorphism $\mathbb{C}[G] \to \mathrm{End}(V_\lambda)$. Also, let χ_λ denote the character of the representation V_λ.

Problem 15. **[10 points]** Write the change of bases formulae for the following three bases for $\mathbb{C}[G]_{\text{class}}$: (i) $\{\delta_C : C \in \text{Conj}(G)\}$, (ii) $\{\chi_\lambda : \lambda \in P(G)\}$, and (iii) $\{P_\lambda : \lambda \in P(G)\}$, where

$$P_\lambda := \frac{\dim(V_\lambda)}{|G|} \sum_{g\in G} \chi_\lambda(g^{-1})\delta_g \in \mathbb{C}[G]_{\text{class}}.$$

Problem 16. **[15 points]** *Plancheral formula.* For $f_1, f_2 \in \mathbb{C}[G]$, show that:

$$\beta(f_1, f_2) = \frac{1}{|G|^2} \sum_{\lambda\in P(G)} \dim(V_\lambda)\text{Tr}(\rho_\lambda(f_1 * f_2)).$$

Problem 17. **[15 points]** Consider the following isomorphisms:

$$\psi : \mathbb{C}[G] \to \bigoplus_{\lambda\in P(G)} \text{End}(V_\lambda), \qquad \varphi : \bigoplus_{\lambda\in P(G)} V_\lambda \otimes V_\lambda^* \to \bigoplus_{\lambda\in P(G)} \text{End}(V_\lambda).$$

(1) Show that ψ and φ are $G \times G$–intertwiners.
(2) For $v \in V_\lambda$ and $\xi \in V_\lambda^*$, consider the following function on G (called *matrix coefficient*): $c_{v,\xi}(g) = \xi(g^{-1} \cdot v)$.

  Show that $\psi^{-1}(\varphi(v \otimes \xi)) = \dfrac{\dim(V_\lambda)}{|G|} c_{v,\xi}$.

Problem 18. **[10 points]** Let G, H be two finite groups. Prove that irreducible, finite–dimensional $G \times H$–representations are precisely $V \otimes W$, where V and W are irreducible, finite–dimensional representations of G and H respectively.

Problem 19. **[20 points]** Compute the character table of S_5.

Problem 20. **[15 points]** Let $f(t) = \displaystyle\sum_{n=0}^{\infty} f_n \frac{t^n}{n!}$ and $g(t) = \displaystyle\sum_{\ell=1}^{\infty} g_\ell t^\ell$ be two formal power series with coefficients from an arbitrary commutative algebra over $\mathbb{Q}$. Note that $g(0) = 0$. Prove that:

$$f(g(t)) = \sum_{n=0}^{\infty} f_n \frac{g(t)^n}{n!} = f_0 + \sum_{N=1}^{\infty} t^N \left(\sum_{k=1}^{N} f_k \left(\sum_{\substack{\lambda\vdash N \\ \ell(\lambda)=k}} \frac{g_\lambda}{\prod_{i\geq 1} \ell_i!} \right)\right) ,$$

where $\ell_i = |\{r : \lambda_r = i\}|$ and $g_\lambda = g_{\lambda_1} \cdots g_{\lambda_k}$.

Problem 21. **[15 points]** Recall the definition of elementary, complete and power sum symmetric polynomials, in N variable. Convention: $e_0 = h_0 = 1$.

$$e_r := \sum_{1\leq i_1<\ldots<i_r\leq N} x_{i_1} \cdots x_{i_r}, \qquad p_r := \sum_{i=1}^{N} x_i^r, \qquad h_r := \sum_{1\leq i_1\leq\ldots\leq i_r\leq N} x_{i_1} \cdots x_{i_r}.$$

Consider the generating series:

$$E(t) = \sum_{r=0}^{N} e_r t^r, \qquad H(t) = \sum_{r=0}^{\infty} h_r t^r, \qquad P(t) = \sum_{n=1}^{\infty} p_n \frac{t^n}{n}.$$

(a) Show that $H(t)E(-t) = 1$ and $H(t) = \exp(P(t))$.
(b) Prove *Newton's identities*:

$$nh_n = \sum_{r=1}^{n} p_r h_{n-r}, \qquad ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}.$$

Problem 22. [**10 points**] Let $n \in \mathbb{Z}_{\geq 2}$ and $\lambda \vdash n$. Let V_λ be the finite–dimensional, irreducible representation of the symmetric group S_n, labelled by λ, as in Frobenius' theorem. Let $w = (1\ 2\ \ldots\ n) \in S_n$. Prove that

$$\text{Trace}(w \text{ acting on } V_\lambda) = \begin{cases} (-1)^k, & \text{if } \lambda = (n-k, 1, \ldots, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Problem 23. [**10 points**] Let $\lambda, \mu \vdash n$. Recall the notation:

$$X_\lambda = \left\{ (I_1, \ldots, I_\ell) : \{1, \ldots, n\} = \bigsqcup_{i=1}^{\ell} I_i \text{ and } |I_j| = \lambda_j, \forall j \right\}$$

(a) Show that X_λ is in bijection with S_n/S_λ, where $S_\lambda = S_{\lambda_1} \times \cdots S_{\lambda_\ell} < S_n$.
(b) Show that there is a bijection between the following three sets:
   (1) $(X_\lambda \times X_\mu)/S_n$,
   (2) $X_\mu^{S_\lambda} := \{a \in X_\mu : w \cdot a = a,\ \forall w \in S_\lambda\}$,
   (3) $\{A = (a_{ij}) \in \mathrm{M}_{n \times n}(\mathbb{Z}_{\geq 0}) : \sum_j a_{ij} = \lambda_i, \forall i \text{ and } \sum_i a_{ij} = \mu_j, \forall j\}$.

Problem 24. [**15 points**] Recall the definition of $(\cdot, \cdot)$ on $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_N]^{S_N}_{\deg = n}$.

(1) For $\lambda \vdash n$, let $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ and $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, where h_r and e_r are defined above (Problem 21). Show that (h_λ, e_μ) is the number of matrices with $0, 1$ entries whose row sums equal λ and column sums equal μ.

$$(h_\lambda, e_\mu) = \left| \left\{ A = (a_{ij}) : a_{ij} \in \{0,1\}\ \forall i, j \text{ and } \sum_j a_{ij} = \lambda_i,\ \forall i; \sum_i a_{ij} = \mu_j,\ \forall j \right\} \right|$$

(2) Let s_λ be the Schur polynomial, and m_μ be the monomial symmetric polynomial. Let $\rho = (N-1, N-2, \ldots, 0)$. Show that:

$$\boxed{(s_\lambda, m_\mu) = \sum_{\substack{w \in S_N: \\ \lambda + \rho - w(\rho) \in S_N \cdot \mu}} \varepsilon(w)}$$

Problem 25. [**10 points**] Let $\mu \vdash (n-1)$. Prove the following (*Pieri's rule*):

$$s_\mu s_{(1)} = \sum s_\lambda\ ,$$

where the sum is over all $\lambda \vdash n$ which can be obtained from μ by adding exactly one box (i.e., $\exists j : \lambda_i = \mu_i, \forall i \neq j$, and $\lambda_j = \mu_j + 1$).

Problem 26. [**10 points**] Let G be a finite group, $H < G$ a subgroup and let $\mathbf{1}$ denote the one dimensional, trivial representation of H. Prove that:

$$\chi_{\mathrm{Ind}_H^G \mathbf{1}}(g) = |(G/H)^g| = \frac{|C \cap H| \cdot |Z_G(g)|}{|H|} ,$$

where $C = \{xgx^{-1} : x \in G\}$ is the conjugacy class containing g, and $Z_G(g) = \{x \in G : xg = gx\}$ is its centralizer.

Problem 27. [**10 points**] Let $\lambda \vdash n$ and let ι_λ is the character of the induced representation U_λ of S_n. Let $\mu \vdash n$ be another partition and let $r_i = |\{j : \mu_j = i\}|$. Use the previous exercise to prove the following formula:

$$\iota_\lambda(\mu) = \sum_{(s_{ij})} \prod_{i \geq 1} \frac{r_i!}{\prod_j s_{ij}!} ,$$

where the sum is over all matrices with $\mathbb{Z}_{\geq 0}$ entries such that $\sum_j s_{ij} = r_i$ and $\sum_i i s_{ij} = \lambda_i$.

Problem 28. [**10 points**] (Reciprocity formula) Let G be a finite group and $H < G$ a subgroup. Let $\phi \in \mathbb{C}[G]_{\text{class}}$ and $\psi \in \mathbb{C}[H]_{\text{class}}$. Prove that

$$\left(\mathrm{Res}_H^G \phi, \psi\right)_H = \left(\phi, \mathrm{Ind}_H^G \psi\right)_G .$$

Problem 29. [**10 points**] Let $m \in \mathbb{Z}_{\geq 2}$. Prove directly that, for any $r \in \mathbb{Z}_{\geq 0}$, $\mathrm{Sym}^r(\mathbb{C}^m)$ is an irreducible representation on $\mathrm{GL}_m(\mathbb{C})$.

Problem 30. [**10 points**] Let $\{R_1, \ldots, R_N\}$ be finite–dimensional representations of a group G. Show that these are pairwise non-isomorphic, irreducible representations of G if, and only if, the matrix coefficients of $R_1, \ldots, R_N$ are linearly independent.

Problem 31. [**15 points**] Recall that, for a graded vector space $U = \oplus_{n=0}^{\infty} U[n]$, such that $\dim(U[n]) < \infty$, for every $n \geq 0$, we define:

$$\text{(Hilbert polynomial)} \qquad P(U; t) = \sum_{n \geq 0} \dim(U[n]) t^n.$$

Let V be a finite-dimensional vector space, and let $T^\bullet(V) = \oplus_{n \geq 0} T^n(V)$, and similarly $\mathrm{Sym}^\bullet(V), \wedge^\bullet(V)$. Prove that:

$$P(T^\bullet(V); t) = \frac{1}{1 - \dim(V) t} , \qquad P(\mathrm{Sym}^\bullet(V); t) = \frac{1}{(1-t)^{\dim(V)}} ,$$

$$P(\wedge^\bullet(V); t) = (1+t)^{\dim(V)} .$$

Problem 32. [**10 points**] Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell)$ be a partition, and let $m \geq \ell$. Show that we have an isomorphism of $\mathrm{GL}_m(\mathbb{C})$–representations:

$$L_{\lambda + (1^\ell)} \cong L_\lambda \otimes \wedge^\ell(\mathbb{C}^m).$$

Problem 33. [**15 points**] *Branching rule for general linear group.* Let λ be a partition of length ℓ, and $m \geq \ell$. Let $L_\lambda^{(m)}$ denote the corresponding representation of $\mathrm{GL}_m(\mathbb{C})$. Consider

$\mathrm{GL}_{m-1}(\mathbb{C})$ as the subgroup of $\mathrm{GL}_m(\mathbb{C})$ via $g \mapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$. Show that:

$$L_\lambda^{(m)}\Big|_{\mathrm{GL}_{m-1}(\mathbb{C})} \cong \bigoplus_\mu L_\mu^{(m-1)} ,$$

where the direct sum is over all $\mu = (\mu_1 \geq \ldots \geq \mu_\ell \geq 0)$ satisfying the interlacing condition:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_\ell \geq \mu_\ell \ .$$

In other words, the Young diagram of μ is contained in the Young diagram of λ and its complement consists entirely of horizontal strips.

Problem 34. [**10 points**] Let $m \geq \mathbb{Z}_{\geq 2}$ and assume that $\{d_n\}_{n=1}^\infty \subset \mathbb{Z}_{\geq 0}$ are defined by the following identity:

$$(1 - mt) = \prod_{n=1}^\infty (1 - t^n)^{d_n} \ .$$

Show that $d_n = \dfrac{1}{n} \sum_{d|n} \mu(d) m^{\frac{n}{d}}$. Here $\mu : \mathbb{Z}_{\geq 1} \to \{0, \pm 1\}$ is the Möbius function defined as: $\mu(1) = 1$, $\mu(r) = 0$ if there is a prime p such that p^2 divides r, and for distinct primes $p_1, \ldots, p_k$, we have $\mu(p_1 \cdots p_k) = (-1)^k$.

Notations: The following problems are about representation theory of $\mathfrak{sl}_2$ (over $\mathbb{C}$). Recall that for every $n \in \mathbb{Z}_{\geq 0}$, L_n denotes the irreducible $n+1$–dimensional representation of $\mathfrak{sl}_2$ described explicitly as follows. L_n has a basis $\{v_0, \ldots, v_n\}$ such that:

$$h \cdot v_\ell = (n - 2\ell) v_\ell, \qquad e \cdot v_\ell = (n - \ell + 1) v_{\ell - 1}, \qquad f \cdot v_\ell = (\ell + 1) v_{\ell + 1},$$

with the understanding that $v_{-1} = v_{n+1} = 0$.

Problem 39. [**10 points**] Prove that $L_n \otimes L_1 \cong L_{n+1} \oplus L_{n-1}$ as $\mathfrak{sl}_2$–representations.

Problem 40. [**50 points**] Exercise 2.15.1 of Pasha's book.

Problem 41. [**30 points**] Exercise 2.16.4 of Pasha's book. Classify all finite–dimensional, irreducible representations of $\mathfrak{sl}_2$ over an algebraically closed field of characteristic $p > 2$.

Problem 42. [**30 points**] Exercise 2.16.5 of Pasha's book.

Problem 43. [**20 points**] Note that the action of e and f on L_n is nilpotent. For any nilpotent operator X, the following is a well–defined invertible operator:

$$\exp(X) = 1 + X + \frac{X^2}{2!} + \ldots$$

Let $\mathsf{s} := \exp(e) \exp(-f) \exp(e)$, considered as a operator on L_n. Show that:

$$\mathsf{s} \cdot v_\ell = (-1)^{n-\ell} v_{n-\ell}.$$

Problem 44. **[20 points]** For any $\lambda \in \mathbb{C}$, define M_λ to be (infinite–dimensional) $\mathfrak{sl}_2$–representation as follows: M_λ has a basis $\{m_r : r \in \mathbb{Z}_{\geq 0}\}$ with $\mathfrak{sl}_2$–action given by (with the understanding that $m_{-1} = 0$):

$$h \cdot m_\ell = (\lambda - 2\ell)m_\ell, \qquad e \cdot m_\ell = (\lambda - \ell + 1)m_{\ell-1}, \qquad f \cdot m_\ell = (\ell + 1)m_{\ell+1}.$$

Show that M_λ is irreducible for $\lambda \notin \mathbb{Z}_{\geq 0}$. For $\lambda \in \mathbb{Z}_{\geq 0}$, show that M_λ has a subrepresentation isomorphic to $M_{-\lambda-2}$, so that we have the following non–split short exact sequence of $\mathfrak{sl}_2$–representations:

$$0 \to M_{-\lambda-2} \to M_\lambda \to L_\lambda \to 0 \ .$$

Problems 45 and 46 are inspired by the following beautiful paper by Etingof and Varchenko: Dynamical Weyl group and applications, Advances in Mathematics, **167** *(2002) pp. 74–127.*

Problem 45. **[30 points]** Let $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$. Let M_λ be the Verma module (see Problem 44) with basis denoted by $\{m_r^{(\lambda)} : r \in \mathbb{Z}_{\geq 0}\}$. Let L_n be the $n+1$–dimensional irreducible representation of $\mathfrak{sl}_2$, with basis denoted by $\{v_0, \ldots, v_n\}$. Let $0 \leq k \leq n$.

Show that, for all but finitely many $\lambda \in \mathbb{C}$, there is a unique $\mathfrak{sl}_2$–intertwiner:

$$\Psi_\lambda^{(n,k)} : M_\lambda \to M_\mu \otimes L_n$$

where $\mu = \lambda - (n-2k)$, whose value on $m_0^{(\lambda)}$ is of the following form, for uniquely determined $c_r \in \mathbb{C}$:

$$\Psi_\lambda^{(n,k)} : m_0^{(\lambda)} \mapsto m_0^{(\mu)} \otimes v_k + \sum_{r=1}^{k} c_r m_r^{(\mu)} \otimes v_{k-r} \ .$$

Deduce that we have the following isomorphism (generically in λ):

$$\mathrm{Hom}_{\mathfrak{sl}_2}(M_\lambda, M_\mu \otimes V) \to V[\lambda - \mu]$$

sending an intertwiner ψ to the coefficient of $m_0^{(\mu)}$ in $\psi(m_0^{(\lambda)})$.

Problem 46. **[30 points]** (continued from Problems 44, 45). Now assume that $\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda \gg 0$. Let n, k and $\Psi_\lambda^{(n,k)}$ be as in Problem 45 above. Show that:

$$\Psi_\lambda^{(n,k)}\left(m_{\lambda+1}^{(\lambda)}\right) = m_{\mu+1}^{(\mu)} \otimes (a(\lambda) v_{n-k}) + \sum_{r \geq 1} b_r(\lambda) m_{\mu+1+r}^{(\mu)} \otimes v_{n-k-r}$$

for some $a(\lambda), b_r(\lambda)$. Compute $a(\lambda)$ explicitly.
