Lecture 0

Outline of the course:

- Asymptotic analysis, integral transforms and applications
- Elliptic integrals and elliptic functions
- Uniformization theorem for Riemann surfaces.

§1. Resummation of an infinite series.

Given a sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$, the sum $\sum_{j=1}^{\infty} a_j$ is defined (Cauchy* 1821) as the limit of its partial sums:

$$\boxed{\begin{array}{l} S = \sum_{j=1}^{\infty} a_j \quad \text{means} \quad S = \lim_{N\to\infty} S_N \\ \text{where} \quad S_N = a_1 + \dots + a_N . \end{array}}$$

We say $\sum_{j=1}^{\infty} a_j$ is convergent (or Cauchy-summable) if $\lim_{N\to\infty} \sum_{j=1}^{N} a_j$ exists. Otherwise, we say $\sum_{j=1}^{\infty} a_j$ is divergent.

Divergent Series whose partial sums remain bounded are often (historically) referred to as oscillatory divergent series.

* Augustin-Louis Cauchy 1789-1857

§2. Example. $1-1+1-1+\dots$

(1) Leibniz[1], around 1696, in a letter to Johann Bernoulli claimed that $\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2}$, on the grounds of "probability".

$$\underbrace{1-1+1-1+\cdots}_{N\text{-terms}} = \begin{cases} 1 & \text{if } N \text{ is odd} \\ 0 & \text{if } N \text{ is even.} \end{cases}$$

(2) Euler, 1743, in a letter to N. Bernoulli, also claimed that the answer is $\frac{1}{2}$. Euler gave two arguments:

- if $x = 1-1+1-1+\cdots$ then $x = 1-x$ hence $x = \frac{1}{2}$.
- $\sum_{n=0}^{\infty}(-\alpha)^n = \frac{1}{1+\alpha}$. Set $\alpha = 1$ to get $\frac{1}{2}$.

Callet's counterargument (1795): Let $m < n$; $m, n \in \mathbb{Z}_{\geq 1}$.

$$\frac{1-x^m}{1-x^n} = \frac{1+x+x^2+\dots+x^{m-1}}{1+x+x^2+\dots+x^{n-1}} \to \frac{m}{n} \text{ as } x \to 1.$$

$$\hookrightarrow = (1-x^m)(1+x^n+x^{2n}+x^{3n}+\dots)$$

$$= 1-x^m+x^n-x^{m+2n}+x^{3n}-\dots$$

So, $1-1+1-1+\dots = \frac{m}{n}$?

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[1] Gottfried Wilhelm von Leibniz 1646-1716

Leonhard Euler 1707-1783.

Remark. - Callet's example has "gaps", and, in fact shows :

$$\frac{m}{n} = 1 + \underbrace{0 + 0 + \dots + 0}_{m-1} + 1 + \underbrace{0 + \dots + 0}_{n-m-1} + 1 \quad \dots$$

§3. Properties of a sum of infinite series :

(G. H. Hardy - Divergent Series)

If we were to generalize the definition of $\sum_{j=1}^{\infty} a_j$, it would be natural to require the following :

Linearity : $\sum_{j=1}^{\infty} a_j \overset{*}{=} S$ & $\sum_{l=1}^{\infty} b_l \overset{*}{=} T$ implies

$$\left(\sum_{j=1}^{\infty} \lambda a_j\right) + \left(\sum_{l=1}^{\infty} \mu b_l\right) \overset{*}{=} \lambda S + \mu T, \quad \forall \lambda, \mu \in \mathbb{C}.$$

Finite reordering : $\sum_{j=1}^{\infty} a_j \overset{*}{=} S \Rightarrow \sum_{l=2}^{\infty} a_l \overset{*}{=} S - a_1$.

Regularity : if $\sum_{j=1}^{\infty} a_j$ is convergent (Cauchy) to S

then $\sum_{j=1}^{\infty} a_j \overset{*}{=} S$.

Example. - Cesàro* summation. $\sum_{j=0}^{\infty} a_j = S$ in the sense of Cesàro

if $\quad S = \lim_{N\to\infty} \frac{1}{N}\left(S_0 + S_1 + \dots + S_{N-1}\right)$

Thus, $\quad 1 - 1 + 1 - 1 + \dots = \frac{1}{2}$ (Cesàro).

More generally, let $k \geq 1$ and define inductively

$$S_n^{(0)} = a_0 + a_1 + \dots + a_n$$

$$S_n^{(k)} = \frac{k}{n+k}\left(S_0^{(k-1)} + \dots + S_n^{(k-1)}\right)$$

(Explicitly, $\displaystyle S_n^{(k)} = \frac{1}{\binom{n+k}{k}} \cdot \sum_{\ell=0}^{n} \binom{\ell+k}{k} a_{n-\ell}$.)

We say $\sum_{j=0}^{\infty} a_j$ is Cesàro-k-summable if $\lim_{n\to\infty} S_n^{(k)}$ exists, and define $\displaystyle \sum_{j=0}^{\infty} a_j \overset{(C,k)}{=} \lim_{n\to\infty} S_n^{(k)}$.

Exercise. - Show that Cesàro resummation is regular.

§4. <u>Abel summation</u> - $\sum_{j=0}^{\infty} a_j$ is <u>Abel summable</u> if the radius of convergence of $\sum_{j=0}^{\infty} a_j x^j$ is ≥ 1 and $\lim_{x\to 1^-} \sum_{j=0}^{\infty} a_j x^j$ exists. In this case $\displaystyle \sum_{j=0}^{\infty} a_j \overset{A}{=} \lim_{x\to 1^-} \sum_{j=0}^{\infty} a_j x^j$.

Ernesto Cesàro 1859-1906; Niels Henrick Abel 1802-1829.

Recall - Abel's theorems :

(1) Given a power series $\sum_{j=0}^{\infty} a_j \cdot x^j$, there exists a unique $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $\sum_{j=0}^{\infty} a_j x^j$ is convergent for $|x| < R$ and divergent for $|x| > R$.

(2) Assume that the radius of convergence of $\sum_{j=0}^{\infty} a_j \cdot x^j$ is 1 and $\sum_{j=0}^{\infty} a_j$ is convergent. Then $\lim_{x \to 1^-} \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_j$ (i.e., Abel summation is regular).

Proof of (1) can be found in Lecture 6 of ACV part 1.

Proof of (2) : let $S_N = a_0 + a_1 + \dots + a_N$ $(N \geq 0)$. Set $S_{-1} = 0$.

Then, $a_n = S_n - S_{n-1}$ and hence

$$\sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} (S_n - S_{n-1}) x^n$$

$$= (1-x) \sum_{n=0}^{N-1} S_n x^n + S_N x^N .$$

We are given that $S = \lim_{N \to \infty} S_N$ exists. So, for $0 < x < 1$, we get

$$\sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{m=0}^{\infty} S_m x^m .$$

To prove : $\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = S.$

Given $\varepsilon > 0$, let $N \gg 0$ be such that $|S_n - S| < \frac{\varepsilon}{2}$ $\forall\, n \geq N.$

Write $$\sum_{n=0}^{\infty} a_n x^n - S = (1-x) \sum_{m=0}^{\infty} S_m x^m - (1-x) \cdot \frac{S}{1-x}$$

$$= (1-x) \sum_{m=0}^{\infty} (S_m - S) x^m$$

$$= \underbrace{(1-x) \sum_{m=0}^{N} (S_m - S) x^m}_{\mathrm{I}} + \underbrace{(1-x) \sum_{m=N+1}^{\infty} (S_m - S) x^m}_{\mathrm{II}}$$

Modulus of II $\leq (1-x) \cdot \frac{\varepsilon}{2} \cdot \sum_{m=N+1}^{\infty} x^m < (1-x) \cdot \frac{\varepsilon}{2} \cdot \frac{1}{1-x} = \frac{\varepsilon}{2}.$

For I, if $C = \sum_{m=0}^{N} |S_m - S|$, then, for $x \in (0,1)$ such that

$1 - x < \frac{\varepsilon}{2C}$, we get :

Modulus of I $\leq \frac{\varepsilon}{2C} \cdot C = \frac{\varepsilon}{2}.$

Hence for $0 < 1 - x < \frac{\varepsilon}{2C}$, $\left| \sum_{n=0}^{\infty} a_n x^n - S \right| < \varepsilon.$ $\square$

§5. Example. Verify that, for $\psi \in (-\pi, \pi)$,

$$\sum_{n=0}^{\infty} (-1)^n e^{in\psi} = \frac{1}{2}\left(1 - i \tan \frac{\psi}{2}\right)$$ in the sense of Abel summation.

Take the real part, and reorder terms to get

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cos(n\psi) = \frac{1}{2}$$

Integrating this relation gives $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(n\psi)}{n} = \frac{\psi}{2}$

One more integral gives $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1-\cos(n\psi)}{n^2} = \frac{\psi^2}{4}.$

$\psi \to \pi$ gives $\displaystyle\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$

§6. <u>Euler resummation</u>. Let $q \in \mathbb{R}_{\geq 0}$ and set

$$a_n(q) = \frac{1}{(q+1)^n} \sum_{\ell=0}^{n} \binom{n}{\ell} q^{\ell} a_{n-\ell} .$$

We say $\displaystyle\sum_{j=0}^{\infty} a_j$ is Euler (q-) summable if $\displaystyle\sum_{n=0}^{\infty} a_n(q)$ is convergent. $\displaystyle\sum_{j=0}^{\infty} a_j \overset{(E,q)}{=} \sum_{j=0}^{\infty} a_j(q)$.

Exercise : if $f(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$

$$f_q(y) = \sum_{n=0}^{\infty} a_n(q)\, y^{n+1} \quad , \quad \text{then}$$

$$f_q(y) = f\left(\frac{y}{q+1-qy}\right).$$

Exercise : geometric series - if $a_n = (-\alpha)^n$ $n = 0, 1, \ldots$

then $a_n(q) = \dfrac{(q-\alpha)^n}{(q+1)^n}$ and

$$\sum_{n=0}^{\infty} a_n(q) \overset{\text{Cauchy}}{=} \frac{1}{1+\alpha} \quad \text{for } |q-\alpha| < q+1.$$

i.e. $\displaystyle\sum_{n=0}^{\infty} (-\alpha)^n = \frac{1}{1+\alpha}$ for $|\alpha| < 1$. Cauchy sense

$$= \frac{1}{1+\alpha} \quad \text{for } |q-\alpha| < q+1. \quad (\text{Euler } q\text{-sense})$$