Lecture 1

Recall - Last time we studied a few methods of resumming an infinite (divergent) series.

§1. Euler (1760) De seriebus divergentibus.

$$\boxed{\sum_{n=0}^{\infty} (-1)^n \cdot n! \approx 0.5963\cdots}$$

Euler's calculation. Let $f(x) = \sum_{n=0}^{\infty} (-1)^n n!\, x^{n+1}$.

Then, $x^2 f' + f = x$.

(Proof: $x^2 f' = \sum_{n=0}^{\infty} (-1)^n (n+1)!\, x^{n+2} = \sum_{m=1}^{\infty} (-1)^{m-1} m!\, x^{m+1}$

$$= -\sum_{m=0}^{\infty} (-1)^m m!\, x^{m+1} + x = -f + x \quad \square)$$

This differential equation can be solved directly to get

$$f(x) = e^{\frac{1}{x}} \int_0^x \frac{e^{-1/t}}{t}\, dt$$

Set $w = \frac{x}{t} - 1$ to get $\boxed{f(x) = \int_0^\infty \frac{e^{-w/x}}{1+w}\, dw}$

Hence it is natural to assign

$$\sum_{n=0}^{\infty} (-1)^n n! = \int_0^{\infty} \frac{e^{-w}}{1+w}\, dw$$

Now $\displaystyle\int_0^{\infty} \frac{e^{-w}}{1+w}\, dw = e \int_1^{\infty} \frac{e^{-t}}{t}\, dt \qquad (t = 1+w)$

$$= e\left(\int_1^{\infty} \frac{e^{-t}}{t}\, dt - \int_0^{1} \frac{1-e^{-t}}{t}\, dt + \int_0^{1} \frac{1-e^{-t}}{t}\, dt \right)$$

Claim*: $\displaystyle\gamma = \int_0^{1} \frac{1-e^{-t}}{t}\, dt - \int_1^{\infty} \frac{e^{-t}}{t}\, dt$

Assuming this, we get :

$$\int_0^{\infty} \frac{e^{-w}}{1+w}\, dw = e\left(-\gamma + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k \cdot k!} \right)$$

(we have written $\displaystyle\frac{1-e^{-t}}{t} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{n-1}}{n!}$

$$\Rightarrow \int_0^{1} \frac{1-e^{-t}}{t}\, dt = \sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{t^n}{n \cdot n!} \right]_{t=0}^{1} . \Bigg)$$

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* Euler-Mascheroni constant $\displaystyle\gamma := \lim_{N\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{N} - \ln(N) \right)$

Using $e = 2.7182818\ldots$ and $\gamma = 0.5772157\ldots$

we get (after 6 terms)

$$\sum_{n=0}^{\infty} (-1)^n \, n! = 0.5963\ldots$$

§2. <u>Proof of</u> $\gamma = \int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^{\infty} \frac{e^{-t}}{t}\,dt$:

First, note that $1 + \frac{1}{2} + \cdots + \frac{1}{n} = \int_0^1 \frac{1-(1-x)^n}{x}\,dx$.

So, $$\int_0^1 \left(1 - \left(1 - \frac{t}{N}\right)^N\right)\frac{dt}{t} = \int_0^{1/N} \left(1 - (1-x)^N\right)\frac{dx}{x} \qquad (x = t/N)$$

$$= \int_0^1 \left(1 - (1-x)^N\right)\frac{dx}{x} - \int_{1/N}^1 \left(1 - (1-x)^N\right)\frac{dx}{x}$$

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \int_{1/N}^1 \frac{dx}{x} + \int_{1/N}^1 (1-x)^N \frac{dx}{x}$$

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \ln(N) + \int_1^N \left(1 - \frac{t}{N}\right)^N \frac{dt}{t}$$

Let $N \to \infty$ to get $$\int_0^1 \frac{1-e^{-t}}{t}\,dt = \gamma + \int_1^{\infty} \frac{e^{-t}}{t}\,dt .$$

§3. Borel* resummation.

Given $a_0, a_1, a_2, \ldots \in \mathbb{C}$, let $f(z) = \sum_{n=0}^{\infty} a_n z^{-n-1}$.

Define $(\mathcal{B}f)(p) = \sum_{n=0}^{\infty} a_n \frac{p^n}{n!}$ (Borel transform of f).

We say $\sum_{n=0}^{\infty} a_n$ is <u>Borel Summable</u> if $\int_0^{\infty} (\mathcal{B}f)(p)\, e^{-p}\, dp$ exists.

$$\sum_{n=0}^{\infty} a_n \overset{B}{=} \int_0^{\infty} (\mathcal{B}f)(p)\, e^{-p}\, dp.$$

Example : (1) geometric series $a_n = (-\alpha)^n$ $n = 0, 1, 2, \ldots$

$$\phi(p) = \sum_{n=0}^{\infty} a_n \frac{p^n}{n!} = e^{-\alpha p}$$

$$\int_0^{\infty} \phi(p)\, e^{-p}\, dp = \int_0^{\infty} e^{-p(1+\alpha)}\, dp = \frac{1}{1+\alpha}$$

for $\mathrm{Re}(\alpha) > -1$.

(2) Factorial series : $a_n = (-1)^n \cdot n!$ $n = 0, 1, 2, \ldots$

$$\phi(p) = \sum_{n=0}^{\infty} a_n \frac{p^n}{n!} = \frac{1}{1+p}.$$

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* Émile Borel 1871 – 1956

So, $$\sum_{n=0}^{\infty} (-1)^n \, n! \overset{\mathcal{B}}{=} \int_0^{\infty} \frac{e^{-p}}{1+p} \, dp .$$

§4. Remarks. - (i) $$\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[p]].$$
$$\sum_{n=0}^{\infty} a_n z^{-n-1} \mapsto \sum_{n=0}^{\infty} a_n \frac{p^n}{n!}$$

The key idea behind Borel resummation is that the "Laplace transform gives an inverse to $\mathcal{B}$".

$$\int_0^{\infty} \frac{p^n}{n!} e^{-pz} \, dp = z^{-n-1} \quad \text{for } \operatorname{Re}(z) > 0 .$$

(ii) Exercise: if $\sum_{n=0}^{\infty} a_n z^{-n-1}$ is convergent in a neighbourhood around ∞, then $\phi(p) = \sum_{n=0}^{\infty} a_n \frac{p^n}{n!}$ has infinite radius of convergence (i.e., gives an entire function of $p \in \mathbb{C}$) and has sub-exponential growth as $p \to \infty$ along any ray emanating from 0.

i.e. $\exists$ M, R, C s.t. $|\phi(p)| < M \cdot e^{C \cdot |p|} \quad \forall \; |p| > R.$

§5. Laplace transform. (along $\mathbb{R}_+$)

$$(\mathcal{L}\phi)(z) := \int_0^\infty \phi(p)\, e^{-pz}\, dp.$$

Lemma. - Assume that $\phi : \mathbb{R}_+ \to \mathbb{C}$ is continuous and satisfies (i) ϕ has at most exponential growth as $p \to \infty$

i.e. $\exists\ M, R, C > 0$ s.t.

$$|\phi(p)| < M \cdot e^{Cp} \quad \forall\ p > R.$$

(ii) $\phi(p)$ has at worst a logarithmic singularity as $p \to 0^+$

i.e. $\exists\ a, b, r > 0$ s.t. $|\phi(p)| < b \cdot p^{a-1}$ $\forall\ p \in (0, r)$.

Then $\mathcal{L}\phi(z) := \int_0^\infty \phi(p) e^{-pz} dp$ defines a holomorphic function of $z \in \mathbb{H}_C = \{\mathrm{Re} > C\}$ (same C as in (i) above).