Lecture 2

Recall : $\quad f(z) = \sum_{n=0}^{\infty} c_n z^{-n-1} \rightsquigarrow \phi(p) = \sum_{n=0}^{\infty} c_n \frac{p^n}{n!} =: (\mathcal{B}f)(p)$

Borel transform.

Laplace transform $\quad (\mathcal{L}\phi)(z) = \int_0^{\infty} \phi(p) e^{-pz} dp.$

§1. Analytic properties of Laplace transforms.

For $\theta \in \mathbb{R} \pmod{2\pi\mathbb{Z}}$ define : $\quad \ell_\theta = \mathbb{R}_{>0} e^{i\theta}$

$C > 0$

$$\mathbb{H}_{\theta,C} = \{x \in \mathbb{C} : \mathrm{Re}(x e^{-i\theta}) > C\}$$

(1) Assume ϕ is a continuous function on $\ell_{-\theta}$ satisfying

- ϕ has at worst a logarithmic singularity as $p \to 0$, $p \in \ell_{-\theta}$

  i.e. $\exists\, m, r, c > 0$ s.t.

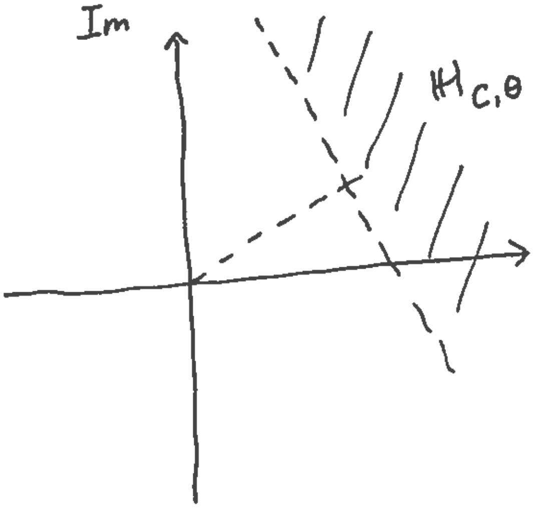
  $$|\phi(t)| < m \cdot |t|^{c-1} \quad \text{for } |t| \in (0,r),\ t \in \ell_{-\theta}.$$

- ϕ has at most an exponential growth as $t \to \infty$ along $\ell_{-\theta}$

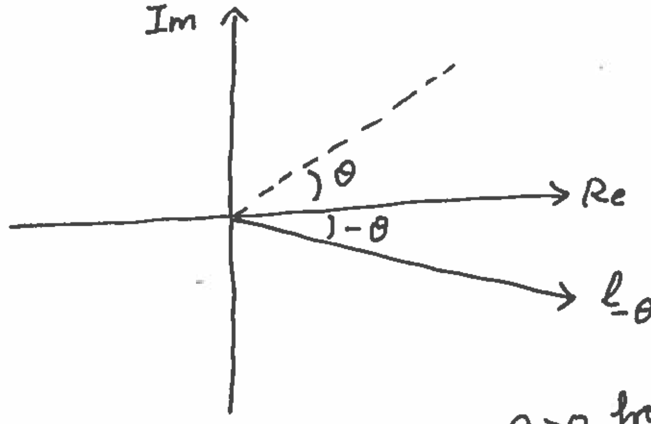
  i.e. $\exists\, M, R, C > 0$ s.t.

  $$|\phi(t)| < M \cdot e^{C\cdot|t|} \quad \text{for } |t| > R,\ t \in \ell_{-\theta}.$$

Define $\quad \mathcal{L}_\theta \phi(z) := \int_{\ell_{-\theta}} \phi(t) e^{-tz} dt \quad$ (Laplace transform along arbitrary ray)



z- plane



Borel plane

$C>0$ from sub-exp. growth of ϕ

Then $\mathcal{L}_\theta \phi\,(z)$ is a holomorphic function of $z \in \mathbb{H}_{\theta,C}$.

<u>Proof.</u>- We need to verify that given a compact set $K \subset \mathbb{H}_C$ and $\varepsilon > 0$,
(take $\theta = 0$ for notational ease)

we can find $\delta > 0$ and $S > 0$ so that

$$\left| \int_0^{\delta_1} \phi(t)\, e^{-zt}\, dt \right| < \varepsilon \quad \text{and} \quad \left| \int_{S_1}^{\infty} \phi(t)\, e^{-zt}\, dt \right| < \varepsilon$$

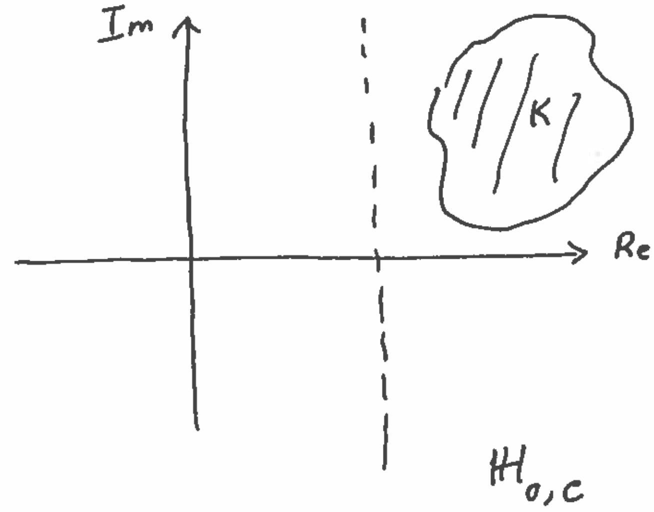
for every $0 < \delta_1 < \delta$, $S_1 > S$, $z \in K$.

Near 0: Choose $\delta < r$ ($r>0$ from singular behaviour of ϕ as $t \to 0$) ($m, c > 0$)

such that $\delta^c < \varepsilon \cdot \frac{c}{m}$.

Then, for every $0 < \delta_1 < \delta$,
$$\left| \int_0^{\delta_1} \phi(t)\, e^{-zt}\, dt \right| < \int_0^{\delta_1} m\, t^{c-1}\, dt = \frac{m \cdot \delta_1^{c}}{c} < \varepsilon.$$

Near ∞: let $M, R, C > 0$ be as in the hypotheses on $\phi(t)$.



Let $D > C$ be such that $\mathrm{Re}(z) > D$, $\forall\, z \in K$.

Choose $S > R$ s.t. $e^{-(D-C)S} < \dfrac{\varepsilon(D-C)}{M}$.

Then, $\forall\, S_1 > S$ and $z \in K$, we have:

$$\left| \int_{S_1}^{\infty} \phi(t)\, e^{-zt}\, dt \right| \leqslant \int_{S_1}^{\infty} |\phi(t)|\, e^{-\mathrm{Re}(z)\, t}\, dt$$

$$< M \int_{S_1}^{\infty} e^{-(D-C)t}\, dt = \frac{M\, e^{-(D-C)S_1}}{D-C} < \varepsilon$$

$\square$

§2. Jump behaviour of Laplace transforms (Stokes' phenomenon)

Assuming $\mathcal{L}_\theta \phi$ makes sense for $\theta \in (\theta_1, \theta_2)$, we will now describe the change in $\mathcal{L}_{\theta_1}\phi$ and $\mathcal{L}_{\theta_2}\phi$.

<u>Mittag-Leffler type assumptions:</u> Assume $\phi : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic, $A \subset \mathbb{C}$ (discrete) set of poles of ϕ.

Assume : $0 \notin A$ and it is possible to find $R_1 < R_2 < \cdots$

so that $\lim_{m \to \infty} R_m = \infty$ and

(i) ϕ has no poles on the circle of radius R_m $(m \geq 1)$.

(ii) ϕ has sub-exponential growth along rays not passing through poles of ϕ, and along $|t| = R_m$ $(m \geq 1)$

i.e. $\exists\ M, R, C > 0$ s.t. $|\phi(t)| < M \cdot e^{C \cdot |t|}$

for every t, $\begin{cases} |t| > R, & t \in \ell_\theta \text{ s.t. } \ell_\theta \cap A = \phi. \\ \text{or } |t| = R_m & (m \geq 1). \end{cases}$

These assumptions make sure that $\mathcal{L}_\theta \phi$ makes sense for $\theta \in \mathbb{R}$ such that $\ell_{-\theta}$ does not pass through poles of ϕ.

The choice of $R_1 < R_2 < \ldots$ is essentially a matter of convenience. We could relax it to :

For every $m \geq 1$, we can find a contour $\mathcal{C}_m \subset \mathbb{C} \setminus A$ s.t.

- $\text{Interior}(\mathcal{C}_1) \subset \text{Interior}(\mathcal{C}_2) \subset \ldots$ $\quad \bigcup_{m=1}^{\infty} \text{Interior}(\mathcal{C}_m) = \mathbb{C}$
- $\forall\ z_0 \in \mathbb{C}$, $\quad \text{distance}(z_0 ; \mathcal{C}_m) \to \infty$ as $m \to \infty$.
- $|\phi(t)| < M e^{C|t|}$ $\quad \forall\ t \in \mathcal{C}_m$ $(m \geq 1)$.

Theorem. - Let $\theta_1 < \theta_2 < \theta_1 + \pi$ be such that

$$\ell_{-\theta_1} \cap A = \ell_{-\theta_2} \cap A = \phi .$$

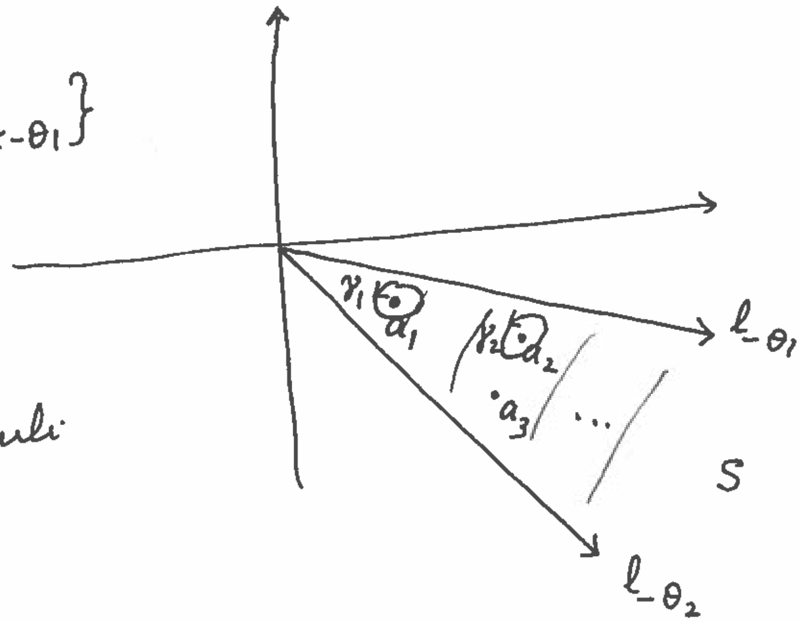
Let $S = \left\{ r \cdot e^{i\psi} \;\middle|\; \begin{array}{l} r > 0 \\ -\theta_2 < \psi < -\theta_1 \end{array} \right\}$

and let

$A \cap S = \{a_1, a_2, a_3, \dots\}$

be arranged in ascending moduli

$$0 < |a_1| \le |a_2| \le \cdots$$



Then $\mathcal{L}_{\theta_2}\phi\,(z) - \mathcal{L}_{\theta_1}\phi\,(z)$

$$= \sum_{K=1}^{\infty} \int_{\gamma_K} \phi(t)\, e^{-zt}\, dt$$

γ_K small enough counterclockwise circle around a_K s.t.

$\overline{\text{Interior}(\gamma_K)} \cap A = \{a_K\}$

$\overline{\text{Interior}(\gamma_K)} \subset S.$

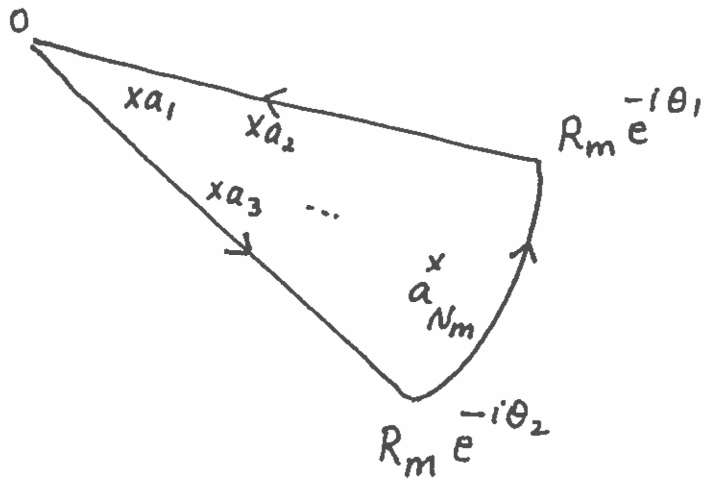
$(\forall\, z \in \mathbb{H}_{\theta_1, C} \cap \mathbb{H}_{\theta_2, C})$

Proof.- Pick $C_1 > C$ and assume $z \in \mathbb{H}_{\theta_1, C} \cap \mathbb{H}_{\theta_2, C}$ be such that $\mathrm{Re}(z e^{-i\psi}) \ge C_1 \quad \forall\, \psi \in (\theta_1, \theta_2)$

(need $|\theta_1 - \theta_2| < \pi$ to be able to do so).

For each $m \geq 1$, let P_m be the following contour

- $0 \to R_m e^{-i\theta_2}$ along $\ell_{-\theta_2}$
- $R_m e^{-i\theta_2}$ to $R_m e^{-i\theta_1}$ along circular arc, say μ_m.
- $R_m e^{-i\theta_1}$ to 0 along $\ell_{-\theta_1}$.



By Cauchy's theorem

$$\int_{P_m} \phi(t)\, e^{-zt}\, dt = \sum_{K=1}^{N_m} \int_{\gamma_K} \phi(t)\, e^{-zt}\, dt$$

Note: left-hand side $= \displaystyle\int_0^{R_m e^{-i\theta_2}} + \int_{\mu_m} - \int_0^{R_m e^{-i\theta_1}}$

So, to prove the theorem, it is enough to show that

$$\int_{\mu_m} \phi(t)\, e^{-zt}\, dt \to 0 \quad \text{as} \quad m \to \infty.$$

By our assumptions,

$$\left| \int_{\mu_m} \phi(t)\, e^{-zt}\, dt \right| < M \cdot e^{-(C_1 - C) R_m} \cdot (\theta_2 - \theta_1) R_m$$

$$\to 0 \quad \text{as} \quad m \to \infty.$$

$\square$

§3. Application to linear differential and difference equations

Prop. – (1) $\mathcal{B}(\partial_z f) = -p\,(\mathcal{B}f)$

Let T_x be the operator of shift by x : $T_x f(z) = f(z+x)$.

(2) $\mathcal{B}(T_x f) = e^{-px}\,\mathcal{B}f$.

Proof. – (1) If $f(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$, then $\partial_z f = -\sum_{n=0}^{\infty} (n+1) c_n z^{-n-2}$

So, $$\mathcal{B}(\partial_z f) = -\sum_{n=0}^{\infty} (n+1) c_n \frac{p^{n+1}}{(n+1)!} = -p \cdot \sum_{n=0}^{\infty} c_n \frac{p^n}{n!}.$$

(2) $$(T_x f)(z) = \sum_{n=0}^{\infty} c_n (z+x)^{-n-1} = \sum_{n=0}^{\infty} c_n z^{-n-1} \underbrace{\left(1+x z^{-1}\right)^{-n-1}}_{\sum_{\ell=0}^{\infty} (-1)^{\ell} \binom{n+\ell}{\ell} x^{\ell} z^{-\ell}}$$

$$= \sum_{N=0}^{\infty} z^{-N-1} \left(\sum_{\ell=0}^{N} (-1)^{\ell} \binom{N}{\ell} x^{\ell} c_{N-\ell} \right)$$

$$\Rightarrow \quad \mathcal{B}(T_x f) = \sum_{N=0}^{\infty} \frac{p^N}{N!} \left(\sum_{\ell=0}^{N} (-1)^{\ell} \binom{N}{\ell} x^{\ell} c_{N-\ell} \right)$$

$$= \sum_{N=0}^{\infty} \left(\sum_{\ell=0}^{N} (-1)^{\ell} \frac{p^{\ell} x^{\ell}}{\ell!} \frac{c_{N-\ell}}{(N-\ell)!} p^{N-\ell} \right)$$

$$= e^{-px} \cdot \sum_{K=0}^{\infty} c_K \frac{p^K}{K!} \qquad \square$$