Lecture 3

Recall:

Formal power series
$$f(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$$

$\xrightarrow{\text{Borel Transform}}$

Formal power series
$$\phi(t) = \mathcal{B}[f](t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

Asymptotic power series (Watson's lemma)

Laplace transform (under some hypotheses on $\phi(t)$)

Holomorphic functions on half plane $\mathbb{H}_{c,\theta} = \{x \in \mathbb{C} : Re(x e^{-i\theta}) > c\}$

$$(\mathcal{L}_\theta \phi)(z) = \int_{\ell_{-\theta}} \phi(t) e^{-zt} dt$$

§1. Definition of asymptotic (power) series (Poincaré*)

Assume $S \subset \mathbb{C}$ is an unbounded open set (i.e., ∞ is adherant to S)

$F(z)$ a holomorphic function defined on S.

We say $F(z) \sim \sum_{n=0}^{\infty} c_n z^{-n-1}$ as $z \to \infty$, $z \in S$ if

for every $N \geq 0$ $$\lim_{\substack{z \to \infty \\ z \in S}} \left(F(z) - \sum_{n=0}^{N-1} c_n z^{-n-1} \right) \cdot z^N \ \cancel{\text{exists}} = 0.$$

If these limits exist for $N = 0, 1, \dots, M$; we say <u>F admits an asymptotic power series expansion to order M</u>.

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* Henri Poincaré (1854-1912) - Sur les intégrales irregulière des équations lineaires (1896)

Examples. (1) If $S = \{x \in \mathbb{C} : |x| > R\}$ (open neighbourhood of ∞)

$F : S \to \mathbb{C}$ holomorphic

and $\lim_{\substack{z \to \infty \\ z \in S}} F(z) = 0$, then F is holomorphic at ∞.

(2) $e^{-z} \sim 0$ as $Re(z) \to \infty$. Thus asymptotic power series do not determine (the) function.

(3) Periodic functions do not admit asymptotic power series expansions (e.g. $\sin(z)$ as $z \to \infty$, $z \in \mathbb{R}$)

§2. Watson's lemma. - Assume $\phi(t)$ is a continuous function of $t \in \mathbb{R}_{>0}$, of sub-exponential growth as $t \to \infty$.

If $\phi(t) \sim \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$ as $t \to 0^+$, then

$$F(z) = (\mathcal{L}_0 \phi)(z) = \int_0^\infty \phi(t)\, e^{-zt}\, dt \sim \sum_{n=0}^{\infty} c_n z^{-n-1} \quad \text{as } Re(z) \to \infty$$

Proof.- To show: for any $N > 0$

$$\lim_{Re(z) \to \infty} z^N \left(F(z) - c_0 z^{-1} - \dots - c_{N-1} z^{-N} \right) = 0 .$$

George N. Watson (1886-1965)

By $\displaystyle\int_0^\infty \frac{t^n}{n!} e^{-zt}\,dt = z^{-n-1}$, we can write

$$\sum_{K=0}^{N-1} c_K z^{-K-1} = \int_0^\infty \left(\sum_{K=0}^{N-1} c_K \frac{t^K}{K!} \right) e^{-zt}\,dt. \quad \text{Hence,}$$

$$F(z) - \sum_{K=0}^{N-1} c_K z^{-K-1} = \int_0^\infty \phi_N(t)\, e^{-zt}\,dt \quad , \quad \text{where}$$

$$\phi_N(t) = \phi(t) - \underbrace{\sum_{K=0}^{N-1} c_K \frac{t^K}{K!}}_{\text{polynomial in } t.}$$

$$\phi_N(t) \sim \sum_{\ell=N}^{\infty} c_\ell \frac{t^\ell}{\ell!} \quad \text{as } t \to 0^+.$$

So, $\displaystyle\lim_{t\to 0^+} \frac{\phi_N(t)}{t^N}$ exists $\left(= \dfrac{c_N}{N!}\right)$, and we can find $A, r > 0$

such that $\quad |\phi_N(t)| < A \cdot t^N \quad , \quad \forall\ t \in (0,r).$

As ϕ has sub-exp. growth, so does ϕ_N, and we have $M, R, C > 0$

such that $\quad |\phi_N(t)| < M \cdot e^{ct} \quad \forall\ t > R.$

Now, we can write

$$\int_0^\infty \phi_N(t)\, e^{-zt}\,dt = \underset{①}{\int_0^r} + \underset{②}{\int_r^R} + \underset{③}{\int_R^\infty} \phi_N(t)\, e^{-zt}\,dt$$

Claim.- ② and ③ are asymptotically zero.

Proof.- ②: $\left| \int_r^R \phi_N(t) e^{-zt} dt \right| \leq m \cdot \int_r^R e^{-xt} dt \qquad x = Re(z)$

$m = Max\{|\phi_N(t)| : r \leq t \leq R\}$

$= m \cdot \dfrac{e^{-rx} - e^{-Rx}}{x} \to 0$ as $x \to \infty$ faster than any polynomial.

Hence $\lim_{x\to\infty} x^N \cdot |②| = 0$ as claimed.

③: $\left| \int_R^\infty \phi_N(t) e^{-zt} dt \right| \leq M \cdot \int_R^\infty e^{(c-x)t} dt = \dfrac{M e^{-(x-c)R}}{x-c} \to 0$ as $x \to \infty$ faster than x^N.

$(x = Re(z))$

□

The remaining integral gives

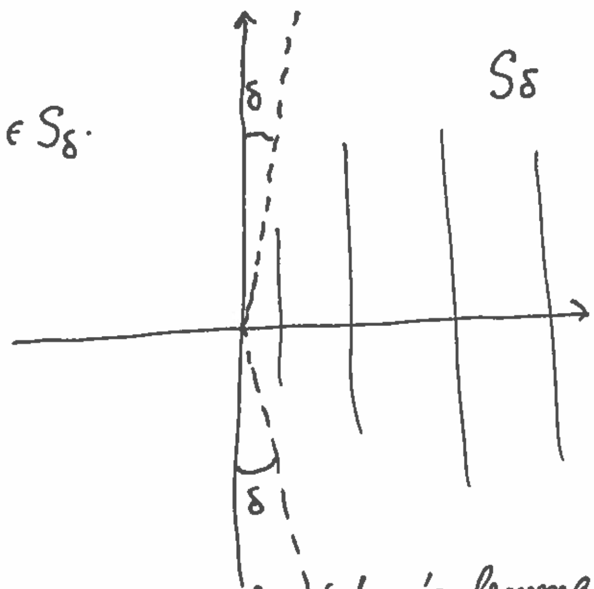
$$\left| \int_0^r \phi_N(t) e^{-zt} dt \right| \leq A \cdot \int_0^r t^N e^{-xt} dt < A \int_0^\infty t^N e^{-xt} dt$$

$$= A \cdot \frac{N!}{x^{N+1}}.$$

Hence, $\lim_{Re(z)\to\infty} \left| z^N \left(F(z) - \sum_{K=0}^{N-1} c_K z^{-K-1} \right) \right| = \lim_{Re(z)\to\infty} \dfrac{A \cdot N! \, |z|^N}{Re(z)^{N+1}} = 0$

□

Remark.- Let $S_\delta = \{ w \in \mathbb{C} : -\frac{\pi}{2} + \delta < \arg(w) < \frac{\pi}{2} - \delta \}$



Then

$$|z| \sin(\delta) \leq Re(z) \leq |z| \; , \forall z \in S_\delta .$$

This implies :

$$\boxed{\begin{array}{l} z \to \infty , \; z \in S_\delta \\ \Longleftrightarrow \quad Re(z) \to \infty \end{array}} .$$

And, therefore, we can write the conclusion of Watson's lemma as :

$$F(z) \sim \sum_{n=0}^{\infty} c_n z^{-n-1} \quad \text{as} \quad z \to \infty , \; z \in S_\delta .$$

§3. Example. -

Consider the difference equation $f(z+1) - f(z) = \frac{-1}{z^2}$.

Method 1 - formal solution : $f(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$.

$$f(z+1) = \sum_{n=0}^{\infty} c_n (1+z)^{-n-1} = \sum_{n=0}^{\infty} c_n z^{-n-1} (1+z^{-1})^{-n-1}$$

$$= \sum_{n=0}^{\infty} c_n z^{-n-1} \left(\sum_{\ell=0}^{\infty} (-1)^\ell \binom{n+\ell}{\ell} z^{-\ell} \right)$$

$$= \sum_{N=0}^{\infty} z^{-N-1} \left(\sum_{\ell=0}^{N} (-1)^\ell \binom{N}{\ell} c_{N-\ell} \right) = -z^{-2} + f(z)$$

Comparing coefficients of z^{-m} $(m = 1, 2, 3, \cdots)$, we get:

$m=1$: $c_0 = c_0$ ✓ $\qquad$ $m=2$: $c_1 - c_0 = c_1 - 1$

$\Rightarrow c_0 = 1.$

$m=3$: $c_2 - 2c_1 + c_0 = c_2$ $\qquad \cdots$

i.e. $c_1 = \frac{1}{2} c_0 = \frac{1}{2}$

$m = N+1$: $\sum_{\ell=1}^{N} (-1)^{\ell} \binom{N}{\ell} c_{N-\ell} = 0$ determines c_{N-1} assuming $c_0, \ldots, c_{N-2}$ are already computed.

Method 2 (direct)

$$f_+(z) = f_+(z+1) + \frac{1}{z^2} = \frac{1}{z^2} + \frac{1}{(z+1)^2} + f_+(z+2)$$

$$\cdots = \sum_{\ell=0}^{\infty} \frac{1}{(z+\ell)^2}$$

Similarly, $f_-(z) = f_-(z-1) - \frac{1}{(z-1)^2} = \frac{-1}{(z-1)^2} - \frac{1}{(z-2)^2} - f_-(z-2)$

We get two solutions

$$f_+(z) = \sum_{\ell=0}^{\infty} \frac{1}{(z+\ell)^2}$$

$$f_-(z) = \sum_{n=1}^{\infty} \frac{-1}{(z-n)^2}$$

[Check: uniform convergence, so

$f_+ : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}$

$f_- : \mathbb{C} \setminus \mathbb{Z}_{\geq 1} \to \mathbb{C}$]

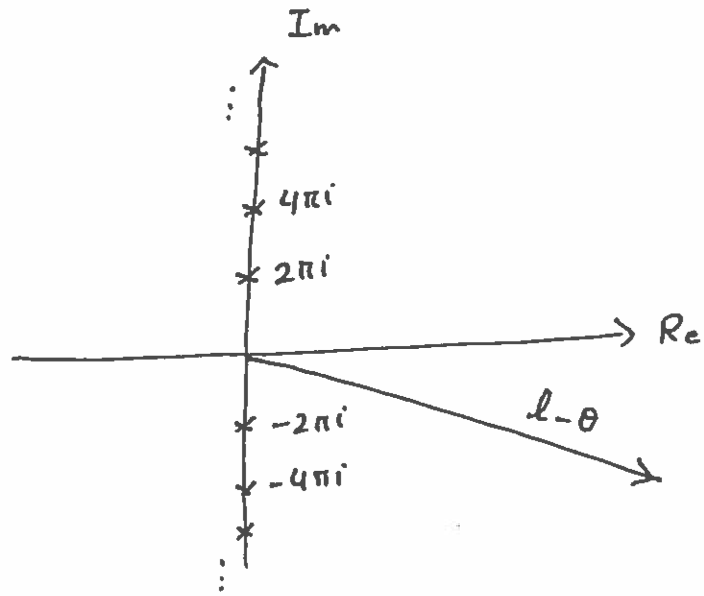
meromorphic fns. on $\mathbb{C}$.

Method 3 (Borel-Laplace)

$$f(z+1) - f(z) = -z^{-2} \underset{\text{Borel transform}}{\rightsquigarrow} (e^{-t} - 1)\phi(t) = -t$$

So, $\phi(t) = \dfrac{t}{1-e^{-t}}$.

$\Rightarrow$ we have **two** solutions.



Poles of $\phi(t)$ at $2\pi i \mathbb{Z}_{\neq 0}$

We can define

$$F_\theta(z) := (\mathcal{L}_\theta \phi)(z)$$

$$= \int_{\ell_{-\theta}} \phi(t)\, e^{-zt}\, dt$$

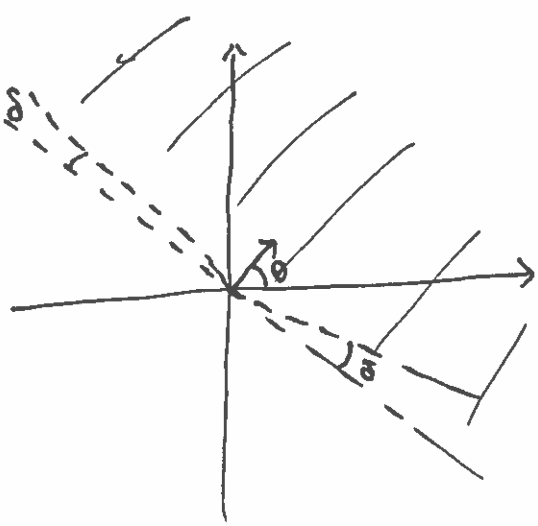
$\left(\theta \not\equiv \pm \frac{\pi}{2} \mod 2\pi\right)$.

Bernoulli numbers.

$$\phi(t) = \frac{t}{1-e^{-t}} = 1 + \frac{1}{2}t + \sum_{\ell=1}^{\infty} B_{2\ell} \frac{t^{2\ell}}{(2\ell)!}$$

By Watson's lemma $\quad F_\theta(z) \sim z^{-1} + \frac{1}{2} z^{-2} + \sum_{\ell=1}^{\infty} B_{2\ell}\, z^{-2\ell-1}$

as $\mathrm{Re}(z e^{-i\theta}) \to \infty$



or as $z \to \infty$, $z \in S_{\theta,\delta}$

$$S_{\theta,\delta} = \left\{ r e^{i\psi} : \begin{array}{l} r > 0 \\ \theta - \frac{\pi}{2} + \delta < \psi < \theta + \frac{\pi}{2} - \delta \end{array} \right\}$$

F_θ is hol. here

Note : $F_{\theta_1}(z) = F_{\theta_2}(z) \quad \forall \; \theta_1, \theta_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$F_{\theta_1}(z) = F_{\theta_2}(z) \quad \forall \; \theta_1, \theta_2 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\{F_\theta\}_{-\frac{\pi}{2}<\theta<\frac{\pi}{2}}$ define one holomorphic function on $\mathbb{C} \setminus \mathbb{R}_{<0}$

call it $F_r(z)$ (r for right)

$\{F_\theta\}_{\frac{\pi}{2}<\theta<\frac{3\pi}{2}}$ ——————————— $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$

call it $F_\ell(z)$

<u>Exercise:</u>

$$F_r(z) - F_\ell(z) = \frac{2\pi i \zeta}{(\zeta-1)^2} = 2\pi i \sum_{\ell=1}^{\infty} \ell \cdot \zeta^{-\ell}$$

$$\left(\zeta = e^{2\pi i z}\right)$$