Lecture 4.

Recall: Laplace transform along direction θ

$$(\mathcal{L}_\theta \phi)(z) = \int_{\ell_\theta = \mathbb{R}_{>0} e^{i\theta}} \phi(t)\, e^{-zt}\, dt$$

Assuming $\phi : \mathbb{C} \dashrightarrow \mathbb{C}$ is meromorphic with discrete set of poles $A \subset \mathbb{C}$ $(0 \notin A)$, and sub-exponential growth at ∞ (so we can take its Laplace transform)

$(\mathcal{L}_\theta \phi)(z)$ makes sense for every θ s.t. $\mathbb{R}_{>0} e^{i\theta} \cap A = \phi$. (empty set)

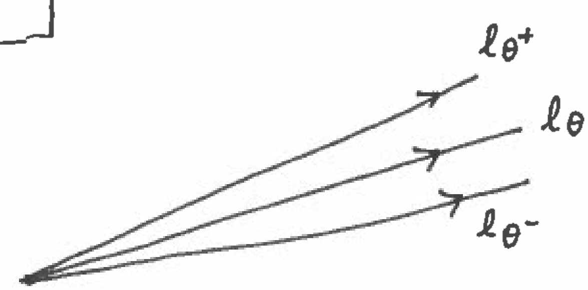
§1. <u>Some definitions</u> .. We say ℓ_θ is a <u>Stokes ray</u> (for ϕ) if $A \cap \ell_\theta \neq \phi$.

<u>Stokes' automorphism</u> S_θ is defined by the following formula

$$\boxed{\mathcal{L}_{\theta^+} \phi = \mathcal{L}_{\theta^-} (S_\theta \phi)}$$

where $\ell_\theta^\pm$ are slightly above/below ℓ_θ s.t. $A \cap \ell_\theta^\pm = \phi$.

(see picture)



Full discontinuity across ℓ_θ is defined by

$$S_\theta = \mathrm{Id} - \mathrm{Disc}_\theta \quad , \text{ so we have}$$

$$\mathcal{L}_{\theta^+} = \mathcal{L}_{\theta^-} \circ (\mathrm{Id} - \mathrm{Disc}_\theta) \quad \text{and hence} \quad \mathcal{L}_{\theta^+} - \mathcal{L}_{\theta^-} = -\mathcal{L}_{\theta^-} \circ \mathrm{Disc}_\theta .$$

The logarithm of Stokes' automorphism defines "alien derivatives" (Écalle - derivation étranger)

$$\boxed{S_\theta = \exp\Big(\sum_{\omega \in A_\theta = A \cap \ell_\theta} e^{-\omega z} \Delta_\omega \Big)} \quad \text{— } (*)$$

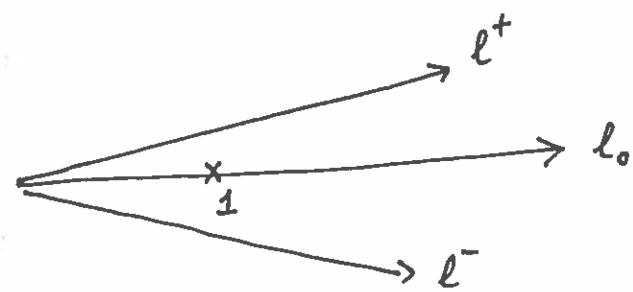
(see §3 below for some remarks)

§2. Example 1. $\phi(t) = \dfrac{1}{1-t}$ $\left(= \text{Borel transform of } f(z) = \sum_{n=0}^{\infty} n!\, z^{-n-1} \right)$

$$S_\theta \phi = \phi \quad \text{for } \theta \neq 0 \bmod 2\pi$$

$$(\mathcal{L}_+ \phi - \mathcal{L}_- \phi)(z)$$

$$= 2\pi i\, e^{-z}$$



Note - both $(\mathcal{L}_\pm \phi)(z)$ solve

$$F'(z) + F(z) = z^{-1}$$

and e^{-z} is a soln. of the associated hgs. equation $F' + F = 0$.

Formula $(*)$, in this case, can be written as

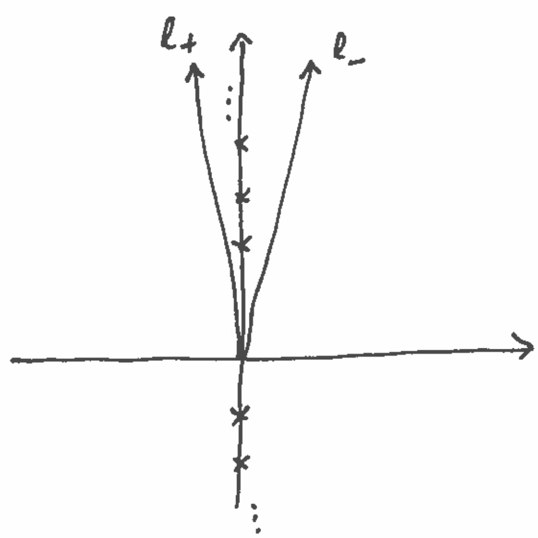
$$\mathcal{L}_{\theta^+}\phi = \mathcal{L}_{\theta^-}\phi + \sum_{k=1}^{\infty} e^{-kz}\,\mathcal{L}_{\theta^-}(\Delta_1^k\cdot\phi)$$

Comparing with $(\mathcal{L}_{\theta^+} - \mathcal{L}_{\theta^-})(\phi) = 2\pi i\, e^{-z}$, we get

$$\Delta_1\left(\frac{1}{1-t}\right) = 2\pi i \qquad \Delta_1^n\left(\frac{1}{1-t}\right) = 0 \quad \forall n \geq 2.$$

Example 2. $\phi(t) = \dfrac{t}{1-e^{-t}}$

$$A = 2\pi i\,\mathbb{Z}_{\neq 0}$$



$$\begin{aligned}(\mathcal{L}_+\phi)(z) - (\mathcal{L}_-\phi)z &= -(2\pi i)^2 \sum_{\ell=1}^{\infty} \ell\cdot e^{-2\pi i \ell z} \\ &= 4\pi^2 \sum_{\ell=1}^{\infty} \ell\cdot e^{-2\pi i\ell z}\end{aligned}$$

Check. – $\Delta_{2\pi i\ell}\cdot\phi = 4\pi^2\ell \quad \forall \ell \geq 1.$

§3. Some remarks on the definition and generalizations.

(i) The definition given above is a bit imprecise. – since terms of the form e^{-wz} are asymptotically 0 and as such cannot be obtained from asymptotic power series analysis alone. It is best to view e^{-wz} – type terms symbolically thus defining $\Delta_\omega \phi$ as a function whose Laplace transform gives the coefficient of e^{-wz} in $\mathrm{Log}(S_\theta)$ – see (*) on page 2 above.

(ii) Even with these caveats, $(\mathcal{L}_0 \phi)(z) = c \in \mathbb{C}$ does not hold for any ϕ. Recall - originally -

$$\begin{tikzcd} z^{-1}\mathbb{C}[[z^{-1}]] \arrow[r, "\mathcal{B}"] & \mathbb{C}[[t]] \arrow[l, bend left, dashed, "\mathcal{L}"] \end{tikzcd}$$

no constant term.

The convention employed in the literature is to introduce a symbol δ s.t. $\mathcal{B}(c) = c\delta$ $\quad (\mathcal{L}\delta = 1)$

(δ = unit for convolution product - to be discussed later)

$$\mathbb{C} + z^{-1}\mathbb{C}[[z^{-1}]] \xrightarrow{\sim} \mathbb{C}\cdot\delta + \mathbb{C}[[t]]$$

Hence, the precise answer from Examples 1, 2 should be

$$\Delta_1\left(\frac{1}{1-t}\right) = 2\pi i \delta$$

$$\Delta_{2\pi i \ell}\left(\frac{t}{1-e^{-t}}\right) = 4\pi^2 \ell \delta \quad (\ell \geq 1)$$

§4. Allowing for log-type singularities in the Borel plane. – review of gamma function.–

$$\int_0^\infty t^{x-1} e^{-t}\, dt = \Gamma(x) \quad \text{for } Re(x) > 0. \qquad (**)$$

is the original Euler's definition of the gamma function.

Here, $t^{x-1} = e^{(x-1)\ln(t)}$ $\quad (t \in \mathbb{R}_{>0})$.

Exercise. – Show that

$$\int_0^\infty t^a e^{-tz}\, dt = \Gamma(a+1)\, z^{-a-1} \qquad \begin{matrix} Re(z) > 0 \\ Re(a) > -1. \end{matrix}$$

Justify differentiation (w.r.t. a) under the integral sign to get

$$\int_0^\infty \ln(t)\, t^a e^{-tz}\, dt = \Gamma'(a+1)\, z^{-a-1} - \log(z)\, \Gamma(a+1)\, z^{-a-1}$$

Set $a = 0$ to get

$$\int_0^\infty \ln(t)\, e^{-tz}\, dt = \left(\Gamma'(1) - \log(z)\right) z^{-a-1}.$$

$\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$.

Thus Laplace transform is defined for more general class of functions (e.g. $t^a = e^{a \log(t)}$: "multi-valued")

The previous exercise allows us to generalize Watson's lemma (see Lecture 3, §2).

If $\phi(t) \sim \sum_{n=1}^{\infty} c_n t^{a_n}$ as $t \to 0^+$ $\qquad -1 < a_1 < a_2 < \cdots$ real numbers

then $$\int_0^\infty \phi(t) e^{-zt} dt \sim \sum_{n=0}^{\infty} c_n \Gamma(a_n+1) z^{-a_n-1}$$ as $Re(z) \to \infty$