Lecture 5

Recall: Laplace transform $(\mathcal{L}_\theta \varphi)(z) := \displaystyle\int_{\mathbb{R}_{>0} e^{i\theta}} \varphi(t)\, e^{-zt}\, dt$

Borel transform $\mathcal{B}\left(z^{-x-1} \cdot \Gamma(x+1)\right) = t^{x}$

Heuristically,

$$\begin{array}{c} \text{Functions in} \\ z\text{-variable} \\ \text{near } \infty \end{array} \underset{\text{Laplace}}{\overset{\text{Borel}}{\rightleftarrows}} \begin{array}{c} \text{Functions in} \\ t\text{-variable} \\ \text{near } 0 \end{array}$$

In applications, we usually have a divergent series in z^{-1} obtained by solving some differential / difference or other functional equation. The scheme depicted above is employed to obtain "actual" - i.e., holomorphic in some sector around ∞ - solutions.

The jump behaviour of these solutions as one varies these sectors (or what is the same thing - rays along which Laplace transform is taken) is refered to as Stokes'* phenomenon.

$$\begin{array}{c} \text{Local Soln.} \\ = \text{formal series in } z \end{array} \overset{\text{Borel-Laplace}}{\rightsquigarrow} \begin{array}{c} \text{Global Solutions} \\ = \text{hol. fns. on sectors +} \\ \text{jump behaviour} \end{array}$$

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* George Stokes (1819 - 1903)

§1. Case of linear differential equations - review:
(see Lectures 9, 10 of ACV 1)

Consider a differential equation $\boxed{\psi'(z) = A(z)\,\psi(z)}$ - (*)

Here $A(z)$ is a matrix-valued meromorphic (usually rational) function of z. $A : \mathbb{C} \dashrightarrow M_{n\times n}(\mathbb{C})$ ($n\times n$ matrices)

<u>Definition</u>.- An isolated singularity $\alpha \in \mathbb{C}$ of $A(z)$ is said to be:

(i) a <u>regular point</u> if α is a removable singularity

(ii) a <u>regular singular (or Fuchsian singularity)</u> point if α is a pole of order 1 (simple pole)

(iii) an <u>irregular singularity</u>, of Poincaré rank r, if α is a pole of order $r \geq 2$.

Frobenius' method for solving (*) amounts to solving it formally and investigating its convergence properties:

<u>Theorem</u>: Assuming $A(z)$ is holomorphic near 0 - i.e.

$$A(z) = A_0 + A_1 z + A_2 z^2 + \dots$$ convergent power series centered at 0, there exists a unique

$$\psi(z) \in \mathbb{C}[[z]],\ \psi(z) = \psi_0 + \psi_1 z + \cdots \ ; \psi_0 = 1.$$

solving (*). This formal solution converges in a neighbourhood of 0.

§2. Regular Singular case: (non-resonant).

Assume $A(z) = \dfrac{\Lambda}{z} + A_0 + A_1 z + \cdots$ near $z = 0$;

and $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is a diagonal matrix s.t.

$$\lambda_i - \lambda_j \notin \mathbb{Z}_{\neq 0} \quad (i \neq j)$$

(non-resonance condition).

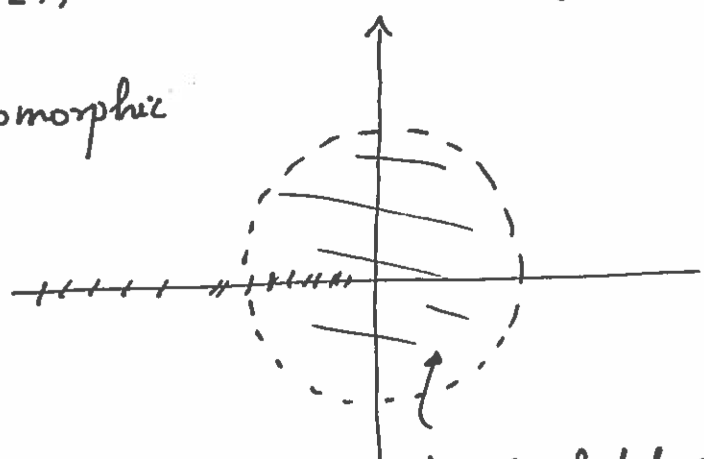
<u>Theorem (Frobenius)</u> - There is a unique formal power series

$$H(z) = H_0 + H_1 z + H_2 z^2 + \cdots \quad ; \quad H_0 = 1$$

such that $H(z) \cdot z^{\Lambda}$ solves $(*)$. This power series
(from page 2)
has a non-zero radius of convergence.

Note: $z^{\Lambda} = \exp(\Lambda \log(z))$ is a multi-valued function.

To understand it as a holomorphic function, one needs to make a cut - say along $\mathbb{R}_{\leq 0}$ to get



domain of definition of $H(z) \cdot z^{\Lambda}$.

$$H(z) \cdot z^{\Lambda} : \underbrace{D(0;R)}_{} \setminus \mathbb{R}_{\leq 0} \to M_{n\times n}(\mathbb{C})$$

disc of radius
R = r.o.c. of $H(z)$

§3. Irregular case – rank 2. (Birkhoff; Subiya; Boalch ...)

Assume: $\quad A(z) = \dfrac{\Lambda}{z^2} + \dfrac{X}{z} + A_0 + A_1 z + \cdots \quad$ near $z=0$.

where $\quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is diagonal matrix with $\lambda_i \neq \lambda_j \;\; \forall\, i \neq j$;

Theorem. – There is a unique power series $Y(z) = \sum_{n=0}^{\infty} Y_n z^n$; $Y_0 = 1$; such that $\;Y(z) \cdot e^{-\Lambda/z} \cdot z^{X_d}\;$ solves $(*)$ – from page 2 above.

Here we write

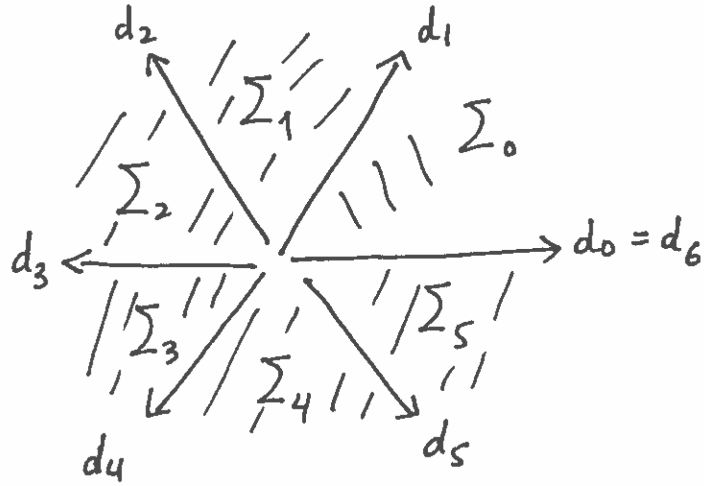
$$X = \underset{\substack{\uparrow \\ \text{diagonal part}}}{X_d} + X_0 \leftarrow \text{off-diagonal part.}$$

Remark. – Unlike regular and regular singular cases, the formal serice obtained here is usually divergent. However, they can be "resummed" in different ways: (see P. P. Boalch "Stokes matrices, Poisson-Lie groups and Frobenius manifolds" – Invent. Math. 2001)

Consider the set of rays $\{(\lambda_i - \lambda_j)\mathbb{R}_{>0} : i \neq j\}$ and let us order them ~~etc~~ in a counterclockwise manner.

$$d_0,\, d_1,\, \ldots,\, d_{2\ell-1},\, d_{2\ell} = d_0$$

$\Sigma_i :=$ sector bounded between d_i and d_{i+1}



Theorem. – $\forall\, i \in \{0, 1, \ldots, 2l-1\}$

$\exists$ a hol. fn. Y_i on Σ_i

s.t. $Y_i(z) \cdot e^{-\Lambda/z}\, z^{X_d}$

solves (*).

$Y_i(z) \sim Y(z)$ (formal series) as $z \to 0$; $z \in \Sigma_i$.

Stokes' matrices encode the relation among these "sectorial" solutions.

e.g. $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad X = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = X_0$ $\quad (X_d = 0)$.

Formal Solution $\quad \begin{bmatrix} 1 & \beta(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-1/z} & 0 \\ 0 & 1 \end{bmatrix}$

$$\beta(z) = \sum_{n=1}^{\infty} (n-1)!\, z^n \leftarrow \text{divergent.}$$

§4. The formalism of Stokes' automorphisms outlined in the last lecture allows us to encode Stokes' matrices in a more systematic way – as well as – it admits generalizations useful for non-linear problems.

$$\begin{array}{ccc} \text{Functions in } z\text{-variable near } z=\infty & \leftrightarrows & \text{Functions in } t\text{-variable near } t=0 \end{array}$$

Let us be a bit informal here and denote these "function spaces" by $\mathcal{F}_z$ and $\mathcal{F}_{t;A}$ ($A \subset \mathbb{C}$ is a discrete set – $\mathcal{F}_{t;A}$ will be functions in t-variable with prescribed singular behaviour at points of A).

"merely a symbol"

From previous lecture :

$$\mathcal{F}_z \;\underset{\mathcal{L}_\theta^-}{\overset{\mathcal{L}_\theta^+}{\leftleftarrows}}\; \sum_{\omega \in A} \left(\mathbb{C}\delta + \mathcal{F}_t\right) \cdot \boxed{e^{-\omega z}}$$

Stokes' auto. S_θ

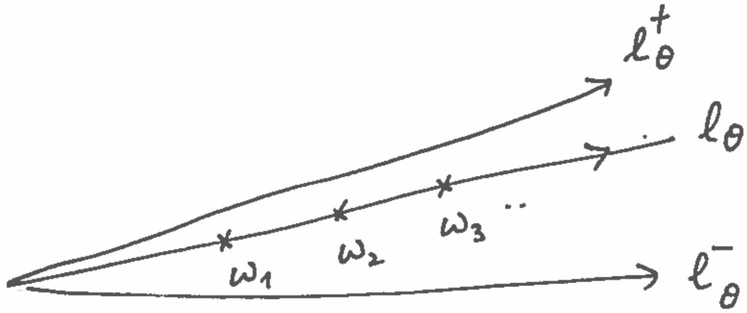
$$\boxed{\mathcal{L}_\theta^+ \varphi = \mathcal{L}_\theta^-(S_\theta \varphi)}$$

see next page.

$(\theta \in \mathbb{R}$ s.t. $A \cap l_\theta \neq \phi)$

and we declare

$$\mathcal{L}(\delta) = 1$$



$$\mathcal{L}\left(\boxed{e^{-\omega z}}\right) = e^{-\omega z}$$

$$A \cap \ell_\theta = \{\omega_1, \omega_2, \omega_3, \dots\}$$

asymptotically 0 as $z \to \infty$ s.t. $\mathrm{Re}(\omega z) > 0$.