Lecture 6

(Formal) Properties of Borel transform

Functions in z-variable "Multiplicative model"	$\xrightarrow{\mathcal{B}}$	Functions in t-variable "Convolutive Model"	
z^{-a-1}		$\dfrac{t^a}{\Gamma(a+1)}$	
$\partial_z f(z)$		$-t\varphi(t)$	$(\varphi = \mathcal{B}f)$
$z \cdot f(z)$		$\partial_t \varphi(t)$	
$f(z+x)$		$e^{-tx} \cdot \varphi(t)$	
$f(\lambda z)$		$\lambda\, \varphi(\lambda^{-1} t)$	$(\lambda \neq 0)$
$f(z) \cdot g(z)$		$(\varphi * \psi)(t)$	$(\psi = \mathcal{B}g)$

§1. Convolution Product

$$t^a * t^b = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}\, t^{a+b+1}$$

Alternately,
$$(\varphi * \psi)(t) = \int_0^t \varphi(x)\, \psi(t-x)\, dx$$

If $\varphi(t) = t^a$ and $\psi(t) = t^b$, then

$$(\varphi * \psi)(t) = \int_0^t x^a (t-x)^b \, dx \quad . \text{ Set } x = t \cdot y$$

$$= \int_0^1 t^{a+b} y^a (1-y)^b \, t \cdot dy = t^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$$

using Euler's "beta function identity"

Remarks. - (1) Convolution product on the "t-side" becomes important when dealing with non-linear problems.

e.g. $\quad f' = -f + \frac{a}{z} + \frac{b}{z} f^2 \quad$ (Riccati equation)

$$\rightsquigarrow \quad (1-t)\,\varphi(t) = a + b \cdot (1 * \varphi * \varphi)$$

$\varphi = \mathcal{B}(f)$

(2) $\quad (\mathbb{C}[\![t]\!], *)$ is a commutative, associative algebra without a unit.

Let $M \subset \mathbb{C}[\![t]\!]$ be the space (linear) of Taylor series expansions of mero. fns. $\mathbb{C} \dashrightarrow \mathbb{C}$ regular at 0.

Then M is not closed under $*$.

e.g. $\quad \varphi = 1$

$\psi(t) = \frac{1}{1-t}$

$$\Rightarrow (\varphi * \psi)(t) = \int_0^t \frac{1}{1-x} \, dx = -\log(1-t) \quad \text{"multivalued"}$$

(3) Propagation of singularities under convolution.

Let us fix a discrete set $A \subset \mathbb{C}$ $(0 \notin A)$ and consider the space of functions which may have "logarithmic singularity" at points of A. Even this space may not be closed under convolution - for general A.

e.g. $\quad \varphi(t) = \psi(t) = \dfrac{1}{1+t} \qquad (A = \{-1\})$.

$$(\varphi * \psi)(t) = \int_0^t \frac{1}{1+x} \frac{1}{1+(t-x)} dx$$

$$= \int_0^t \left(\frac{1}{1+t-x} + \frac{1}{1+x} \right) \frac{dx}{t+2}$$

$$= \frac{1}{t+2} \Big[\log(1+x) - \log(1+t-x) \Big]_0^t$$

$$= \frac{2 \log(1+t)}{2+t} \quad \text{has another singularity at } t = -2$$

Ex. $\quad \varphi = \dfrac{1}{1+t}$ as above. Show that

$$(\varphi * \varphi * \varphi)(t) = \begin{pmatrix} \dfrac{\pi^2}{12} + 2 \log(1+t) \log(2+t) \\ + \operatorname{Li}_2\left(\dfrac{1}{t+2}\right) + \operatorname{Li}_2(-t-1) - \operatorname{Li}_2\left(\dfrac{t+1}{t+2}\right) \end{pmatrix} \cdot \frac{2}{t+3}$$

where $\quad \operatorname{Li}_2(u) = - \displaystyle\int_0^u \frac{\log(1-x)}{x} dx = \sum_{k=1}^{\infty} \frac{u^k}{k^2}$.

§2. Endlessly continuable germs and (simple) resurgent functions.

Let $A \subset \mathbb{C}$ be a discrete set. Assume $0 \in A$.

Define Σ_A = set of homotopy classes of paths $\gamma : [0,1] \to \mathbb{C}$ s.t. $\gamma(0) = 0$ (start at 0)

$$\gamma(t) \in \mathbb{C} \setminus A \quad \forall\, t \in (0,1].$$

(we adjoint a point $\hat{0}$ to this set).

$$\Sigma_A = \{\hat{0}\} \cup \frac{\{\gamma : [0,1] \to \mathbb{C} \text{ s.t. } \gamma(0)=0 \text{ and } \gamma(t) \notin A \ \forall\, t \in (0,1]\}}{\gamma_1 \sim \gamma_2 \text{ if there is a homotopy between } \gamma_1 \,\&\, \gamma_2 \text{ avoiding } A.}$$

$$\Big\downarrow \pi$$

$$(\mathbb{C} \setminus A) \cup \{0\} \qquad \pi(\hat{0}) = 0 \quad \text{and} \quad \pi([\gamma]) = \gamma(1).$$

Note: Σ_A has a structure of Riemann surface with basic open neighbourhoods given as follows. For γ a path with $p = \gamma(1)$ and $r > 0$ s.t. $D(p;r) \subset \mathbb{C} \setminus A$ $\quad (D(p;r) = \{z : |z-p| < r\})$

$N(\gamma; r)$ = set of paths $0 \xrightarrow{\gamma} p \longrightarrow q$, where $q \in D(p;r)$ and $p \to q$ is a straight line segment

Under $\pi : \Sigma_A \to (\mathbb{C} \setminus A) \cup \{0\}$, $\quad N(\gamma; r) \cong D(\gamma(1); r)$

A power series $\varphi(t)$ near $t = 0$ of non-zero radius of convergence is said to be "endlessly continuable" on $\mathbb{C} \setminus A$ if

φ lifts to a holomorphic function

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{F_\varphi} & \mathbb{C} \\ \downarrow \pi & & \\ \mathbb{C} & & \end{array}$$

s.t. $F_\varphi \big|_{D(\hat{0};r)} : D(\hat{0};r) \to \mathbb{C}$

$$\begin{array}{ccc} D(\hat{0};r) & \longrightarrow & \mathbb{C} \\ \cong \downarrow \pi & \nearrow \varphi & \\ D(0;r) & & \end{array}$$

(so, F_φ restricted to $D(\hat{0};r) \cong D(0;r)$ is same as φ.)

A power series $f(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ is called <u>resurgent</u> if its Borel tranform is endlessly continuable (for some discrete set $A \subset \mathbb{C}$) and has sub-exponential growth.

(i.e. $\exists\ M, C > 0$ s.t. $|c_n| < M \cdot C^n \cdot n!$ $\forall n$

\- Gevrey 1-series)

Examples. - (1) $\varphi(t) = \sum_{k=0}^{\infty} \frac{t^k}{k+1} = -\frac{\log(1-t)}{t}$

is an endlessly continuable germ ($A = \{0, 1\}$).

(2) $\varphi(t) = \sum_{\ell=0}^{\infty} t^{2^\ell} = t + t^2 + t^4 + t^8 + \dots$

is <u>NOT</u> endlessly continuable - since $\varphi(t)$ is divergent for $\{t = e^{2\pi i k/2^n} : k \text{ odd},\ n \geq 0\}$ - i.e. the circle (unit) is a <u>natural boundary</u> beyond which φ cannot be analytically continued

Theorem. – Assume $A \subset \mathbb{C}$ is a discrete set as before.

Then, the space of endlessly continuable germs (rel. to A) is closed under convolution if, and only if, A is closed under addition (i.e $a, b \in A \Rightarrow a+b \in A$).

That the condition is necessary follows from the same calculation as on page 3 above:

$$\frac{1}{t-a} * \frac{1}{t-b} = \frac{1}{t-(a+b)}\left(\log\left(1-\frac{t}{a}\right) + \log\left(1-\frac{t}{b}\right)\right)$$

Sufficiency boils down to the following topological fact:

If $A \subset \mathbb{C}$ is a discrete set, closed under addition and γ is a path in $\mathbb{C} \setminus A$, $C_0 : 0 \to \gamma(0)$ straight ray avoiding A, then $\exists$ a smooth family of symmetrical* paths

$$C_s : 0 \longrightarrow \gamma(s) \qquad C_s\big|_{s=0} = C_0$$

and each C_s is in $\mathbb{C} \setminus A$.

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* C: path from 0 to ζ is symmetric if $C(1-x) = \zeta - C(x)$ $\forall\, x \in [0,1]$