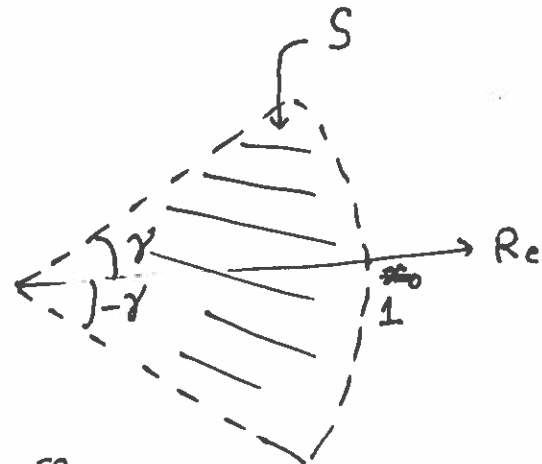
§1. Borel-Ritt lemma.- Given an arbitrary power series $\sum_{n=0}^{\infty} c_n z^{-n-1}$ and a sector $S_R = \{ z \in \mathbb{C} : |z| > R \,;\, \alpha < \arg(z) < \beta \}$, there exists a holomorphic function $F : S_R \to \mathbb{C}$ s.t.

$$F(z) \sim \sum_{n=0}^{\infty} c_n z^{-n-1} \quad \text{as} \quad z \to \infty, \; z \in S_R.$$

Proof.- For simplicity we work near 0, $x = z^{-1}$, and assume that the given sector is bisected by $\mathbb{R}_+$; with radius 1.

Set
$$b_n = \begin{cases} \dfrac{1}{|c_n|} & \text{if } c_n \neq 0 \\ 0 & \text{if } c_n = 0 \end{cases}$$



Define
$$\alpha_n(x) = 1 - e^{-b_n / x^{\beta}}$$

where $0 < \beta < 1$ is small enough so that $\mathrm{Re}(b_n x^{-\beta}) > 0$ for every $x \in S$.

Note : $|1 - e^{y}| < |y|$ for $\mathrm{Re}(y) < 0$. (Check.)

So, for each $n \geq 0$, $|c_n \alpha_n(x) x^n| \leq |c_n| \, |b_n| \, |x|^{n-\beta} \;\; \forall x \in S$.

Hence $F(x) = \sum_{n=0}^{\infty} c_n \alpha_n(x) x^n$ is majorized by

$$\sum_{n=0}^{\infty} |x|^{n-\beta}$$ which converges uniformly for $|x| \leq |x_0| < 1$.

We now replace the given formal series

$$\sum_{n=0}^{\infty} c_n x^n \rightsquigarrow \sum_{n=0}^{\infty} c_n \underbrace{\alpha_n(x)}_{\text{"convergence factors"}} x^n =: F(x)$$

We have shown that $F(x)$ converges uniformly rel. to cpct subsets of S, hence defines a holomorphic function $S \to \mathbb{C}$.

It remain to check that $F(x)$ is asymptotic to $\sum_{n=0}^{\infty} c_n x^n$.

Let $m \geq 0$; and write.

$$x^{-m}\left(F(x) - \sum_{n=0}^{m} c_n x^n \right) = -\sum_{m=0}^{m} c_n e^{-b_n/x^{\beta}} x^{-m+n} + \sum_{n=m+1}^{\infty} c_n \alpha_n(x) x^{-m+n}$$

The first term on the right-hand side goes to 0 as $x \to 0$ in S since exponential dominates polynomials:

$$\lim_{x \to 0^+} e^{-\frac{b}{x}} \cdot x^{-k} = 0$$

The second term can be bounded as

$$\left| \sum_{n=m+1}^{\infty} c_n \alpha_n(x) x^{-m+n} \right| \leq \sum_{n=m+1}^{\infty} |x|^{n-m-\beta} < \frac{|x|^{1-\beta}}{1-|x|} \xrightarrow[\text{as } x \to 0]{} 0$$

Hence $$\lim_{\substack{x \to 0 \\ x \in S}} x^{-m}\left(F(x) - \sum_{n=0}^{m} c_n x^n\right) = 0, \quad \forall\, m \geq 0$$

$\square$

Remark - The angular opening of the sector in Borel-Ritt lemma could be $\geq 2\pi$, provided one thinks of it as a sector in the Riemann Surface $\widetilde{\mathbb{C}\setminus\{0\}}$.

§2. Analogy with summability problem. - Let us keep a domain containing ∞ as its adherant point fixed - say some right-half plane $\mathbb{H}$. We want a well-defined operator "resummation" $\mathcal{R} : \mathbb{C}[[z^{-1}]] \longrightarrow Hol(\mathbb{H})$ (holomorphic functions)

satisfying:

(0) $\mathcal{R}$ is linear: $\quad \mathcal{R}(af(z)+bg(z)) = a\,\mathcal{R}(f(z)) + b\,\mathcal{R}(g(z))$
$\forall\, a,b \in \mathbb{C}$; $f,g \in \mathbb{C}[[z^{-1}]]$.

(1) Regularity: if $f(z)$ is the Taylor series expansion of a holomorphic function $F : \{z : |z|>R\} \to \mathbb{C}$ near ∞, then $\mathcal{R}f = F$.

(2) Asymptotics: $\quad (\mathcal{R}f)(z) \sim f(z)$ as $Re(z) \to \infty$.

(3) Functional prop: $\quad \mathcal{R}$ commutes with ∂_z and T_x.
shift operator

(4) (For applications to non-linear problems)

$$\mathcal{R}(f(z)\,g(z)) = \mathcal{R}(f(z)) \cdot \mathcal{R}(g(z))$$

- i.e., $\mathcal{R}$ is an algebra homomorphism.

(5) Reality condition : if $f(z) \in \mathbb{R}[[z^{-1}]]$, then

$$(\mathcal{R}f)(\bar{z}) = \overline{(\mathcal{R}f)(z)} \qquad ((\mathcal{R}f)(x) \in \mathbb{R} \text{ for } x \in \mathbb{R}).$$

In the previous lecture, we singled out a subalgebra of <u>resurgent power series</u> $\mathcal{S} \subset \mathbb{C}[[z^{-1}]]$ where the Borel-Laplace method gave us a well-defined resummation map

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl, "\mathcal{B}"'] \arrow[dr, "\mathcal{R}_\theta = \mathcal{L}_\theta \circ \mathcal{B}"] & \\
\mathbb{C}\delta \oplus \mathcal{E}_{(t)} \arrow[rr, "\mathcal{L}_\theta"] & & \mathrm{Hol}(\mathbb{H}_\theta)
\end{tikzcd} \qquad - (*)$$

$(\mathcal{E}_{(t)} \subset \mathbb{C}[[t]])$

Loosely speaking, $\mathcal{S}$ consists of power series whose Borel transform is "amenable to Laplace transforms".

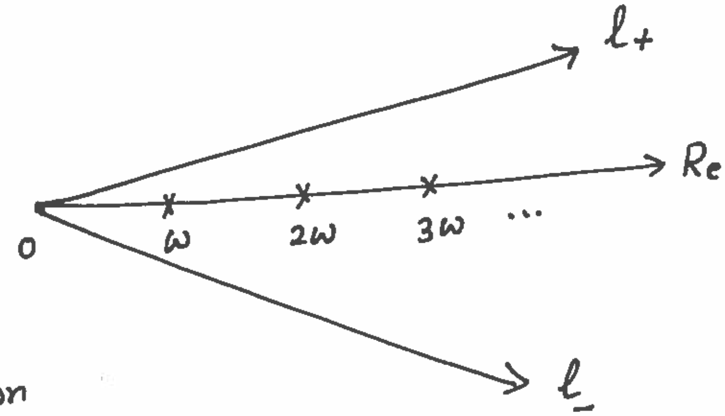
For definiteness, let us keep $\theta = 0$ in $(*)$. Let $A \subset \mathbb{C}$ be a discrete set and $\mathcal{E}^{(A)} \subset \mathbb{C}[[t]]$ consist of endlessly continuable germs rel. to A - of subexponential growth as $t \to \infty$.

$$\mathcal{S} = \left\{ c + \sum_{n=0}^{\infty} c_n z^{-n-1} \in \mathbb{C}[[z^{-1}]] \text{ such that } \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \in \mathcal{E}^{(A)} \right\}.$$

If $A \cap \mathbb{R}_+ = \phi$, then $\mathcal{R} = \mathcal{L}_0 \circ \mathcal{B}$ is a well-defined resummation operator - satisfying all the conditions listed above. Let us discuss the case when we have singularities along $\mathbb{R}_+$

§3. Let $\omega \in \mathbb{R}_{>0}$ and assume that

$$A = \mathbb{Z}_{\geq 1}\omega \subset \mathbb{R}_+$$



The two lateral resummation maps fail to satisfy the reality condition.

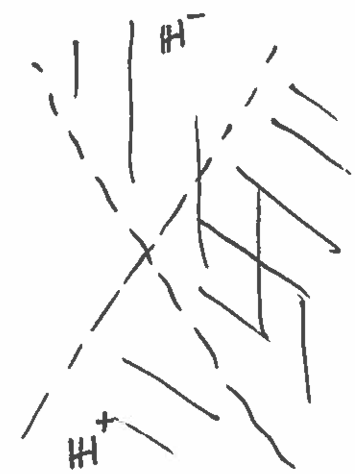
The idea here is to take a "well-balanced average" of the two.

$$\mathcal{R}^{\pm} = \mathcal{L}^{\pm} \circ \mathcal{B} : \mathbb{S} \longrightarrow Hol(\mathbb{H}^{\pm})$$

As $\mathrm{Re}\, z \to \infty$, $\mathcal{R}^{\pm} f$ is asymptotic to f and

$$\mathcal{R}^{\pm} f = f + \sum_{\ell=1}^{\infty} f_\ell^{\pm} \cdot e^{-\ell \omega z}$$

(Trans-series expansion)



We extend $\mathcal{R}^{\pm}$ to automorphisms of $\mathcal{S}[\![e^{-\omega z}]\!]$

$$\mathcal{R}^{\pm}\left(\sum_{\ell=0}^{\infty} f_\ell(z)\, e^{-\ell\omega z}\right) = \sum_{\ell=0}^{\infty} \mathcal{R}^{\pm} f_\ell \cdot e^{-\ell\omega z}.$$

Note - both $\mathcal{R}^{\pm}$ are identity modulo $e^{-\omega z}$.

Stokes' automorphism relates the two lateral resummations

$$\boxed{\mathcal{R}^{+} f = \mathcal{R}^{-}(\mathcal{S}(f))}$$

Again $\mathcal{S}(f) = f \mod e^{-\omega z}$,

so we can take its logarithm and hence fractional powers.

Fact: Logarithm of an algebra automorphism (id mod max. ideal...) is a derivation.

Thus, the following formula defines derivations (strange/alien)

$$\mathcal{S}(f) = \exp\left(\sum_{\ell=\phi}^{\infty} e^{-\ell\omega z} \Delta_\ell\right) \cdot f$$

<u>Median resummation</u> is defined by

$$\begin{aligned} \mathcal{R}(f) &= \mathcal{R}_{-}\left(\mathcal{S}^{-1/2}(f)\right) \\ &= \mathcal{R}_{+}\left(\mathcal{S}^{1/2}(f)\right) \end{aligned}$$

(Check: the reality condition holds).