Lecture 8

Recall - definition of resummation operator and Stokes' automorphism:

$$\mathbb{C}[\![z^{-1}]\!] \underset{\mathcal{B}^{-1}}{\overset{\mathcal{B}}{\rightleftarrows}} \mathbb{C}\delta \oplus \mathbb{C}[\![t]\!]$$

vector space iso (actually alg. iso.)

$A \subset \mathbb{C}$ discrete subset. We identify a good set of "endlessly continuable germs (relative to A)" with sub-exponential growth*, say $\mathcal{C}^{(A)} \subset \mathbb{C}[\![t]\!]$ - and let $M \subset \mathbb{C}[\![z^{-1}]\!]$ be such that $\mathcal{B}(M) = \mathbb{C}\delta + \mathcal{C}^{(A)}$.

The resummation operator along a direction θ is then the composition $\mathcal{R}_\theta = \mathcal{L}_\theta \circ \mathcal{B} : M \longrightarrow \mathcal{H}ol(\mathbb{H}_{-\theta})$. This operator makes sense if $l_\theta = \mathbb{R}_{>0} e^{i\theta}$ is <u>not</u> a Stokes' ray, i.e. $l_\theta \cap A = \phi$.

If l_θ is a Stokes' ray, we introduced Stokes' automorphism $\mathcal{S}_\theta$ by the formula $\mathcal{R}_{\theta^+} = \mathcal{R}_{\theta^-} \circ \mathcal{S}_\theta$.

<u>Note</u>: for the next discussion, we assume that each point $a \in A$ is a simple singularity of $\varphi \in \mathcal{C}^{(A)}$, i.e. near a, φ has the form $\dfrac{\alpha}{t-a} + \cancel{\beta} \cdot \lambda_a(t-a) \log(t-a) + \rho_a(t-a)$

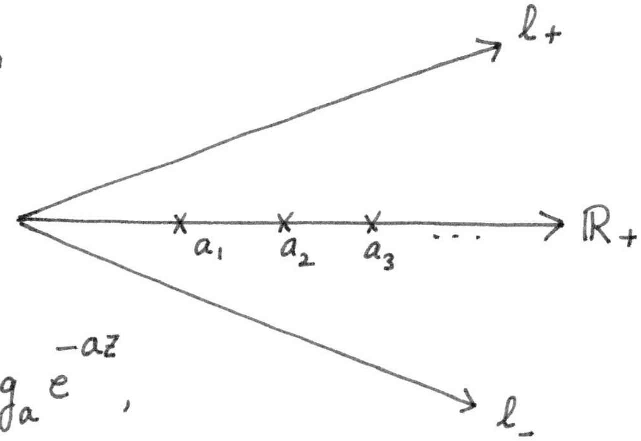
where $\alpha \in \mathbb{C}$ and λ_a, ρ_a are convergent power series.

§1. For definiteness, let $\theta = 0$ and $A \subset \mathbb{R}_+$ be a discrete semigroup (i.e. $a, b \in A \Rightarrow a + b \in A$).

Then, we can formally extend the resummation operators to the algebra of (height 1, log-free) trans-series

$$T := M[\![e^{-az} : a \in A]\!]$$

Thus, the Stokes' automorphism



$$\mathcal{S}_0(f) = f + \sum_{a \in A} f_a e^{-az},$$

where, if $\mathcal{R}_+ f - \mathcal{R}_- f = \sum_{a \in A} g_a e^{-az}$,

then f_a is given by $\mathcal{R}_-(f_a) = g_a$.

Note: $\mathcal{S}_0 \circlearrowright T = M[\![e^{-az} : a \in A]\!]$ is identity modulo

$$\left[\left(\mathcal{S}_0 \left(\sum_{a \in A \cup \{0\}} h_a \cdot e^{-az} \right) \right) = \sum_{a \in A \cup \{0\}} \mathcal{S}_0(h_a) e^{-az} \right]$$

exponentially decaying terms e^{-az} $(a \in A)$ as $Re(z) \to \infty$.

As resummation operators commute with algebraic operations, $\mathcal{S}_0$ is an algebra automorphism.

Exercise - Logarithm of an algebra automorphism is a derivation.

<u>Alien (étrange) derivations</u> $\Delta_a : M \to M$ $(a \in A)$ are defined by $\mathcal{S}_0 = \exp\left(\sum_{a \in A} e^{-az} \Delta_a\right)$, i.e. $\forall f \in M$, we have

$$\mathcal{S}_0(f) = \left(Id + \left(\sum_{a \in A} e^{-az} \Delta_a\right) + \frac{1}{2}\left(\sum_{a \in A} e^{-az} \Delta_a\right)^2 + \cdots\right)(f)$$

$$= f + \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{a_1, \ldots, a_K \in A} e^{-(a_1 + \cdots + a_K)z} \Delta_{a_1} \cdots \Delta_{a_K}(f)$$

<u>Main properties</u> (1) $\Delta_a(f_1 f_2) = \Delta_a(f_1) f_2 + f_1 \Delta_a(f_2)$

(2) $\Delta_a(\partial_z f) = \partial_z(\Delta_a f) - a(\Delta_a f)$

i.e. $e^{-az} \Delta_a$ commutes with ∂_z.

§2. Non-linear 1^{st} order differential equation.

$$f'(z) = b_0(z) + b_1(z) f(z) + b_2(z) f(z)^2 + \cdots \qquad -(*)$$

Assume that each $b_n(z)$ is holomorphic near ∞, $b_1(z) = a + z^{-2}(\cdots)$ $(a \in \mathbb{R}_{>0})$

and $b_n(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$.

e.g. $$f'(z) = -z^{-1} + a f(z) - b z^{-1} f(z)^2 \qquad a \in \mathbb{R}_{>0},\ b \in \mathbb{C}. \qquad -(**)$$

One can easily show that, under the assumptions imposed on $b_n(z)$'s,

(*) has a unique formal solution $f(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$.

e.g. setting $f(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ in (**) gives

$$a\left(\sum_{n=0}^{\infty} c_n z^{-n-1}\right) = z^{-1} + b z^{-1}\left(\sum_{\ell=0}^{\infty} c_\ell z^{-\ell-1}\right)^2 - \sum_{m=0}^{\infty}(m+1) c_m z^{-m-2}$$

Comparing coefficients of z^{-N-1} $(N \geq 0)$, we get c_n's recursively:

$N=0$: $a c_0 = 1$. $\qquad N=2$: $a c_2 = b c_0^2 - 2c_1$

$N=1$: $a c_1 = -c_0$ $\qquad$ and so on.

Locating poles / singularities of $\varphi(t) = \mathcal{B}(f)$.

$\varphi(t)$ has simple singularities along $\{-a, -2a, -3a, \ldots\}$

This is because $(*)$ can be re-written in the form

$$(*') : \quad a f(z) - f'(z) = \sum_{n=0}^{\infty} a_n(z) f(z)^n \qquad a_n(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$$

which, for $\varphi(t) = \mathcal{B}(f)$ becomes (let $\alpha_n(t) = \mathcal{B}(a_n)$ - entire fns.

$$(t+a)\varphi(t) = \sum_{n=0}^{\infty} \alpha_n(t) * \underbrace{\varphi(t) * \cdots * \varphi(t)}_{n\text{-terms}}$$

So, $\varphi(t)$ has simple singularity at $-a$ and its subsequent shifts $-2a, -3a, \ldots$ by the general properties of convolution.

<u>Trans-series ansatz</u> - Let $F(z;\sigma) = \sum_{n=0}^{\infty} f_n(z) e^{naz} \sigma^n$ be a solution of $(*')$; where $f_n(z) \in \cancel{z}\,\mathbb{C}[[z^{-1}]]$; $f_0(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$.

So, by $\partial_z F(z;\sigma) = \sum_{n=0}^{\infty} \sigma^n e^{naz} (na + \partial_z)\cdot f_n(z)$, we get

$$\sum_{n=0}^{\infty} \sigma^n e^{naz} ((n-1)a + \partial_z)\cdot f_n(z) = -\sum_{m=0}^{\infty} a_m(z) F(z;\sigma)^m$$

Again, comparing coefficients, we find :

$n=0$: $\quad (-a + \partial_z) f_0(z) = -\sum_{m=0}^{\infty} a_m(z) f_0(z)^m$ - i.e. $f_0(z) = f(z)$ our original soln.

$n=1$: $\quad \partial_z f_1 = \left(-\sum_{m=1}^{\infty} a_m(z)\, m . f_0^{m-1}\right) \cdot f_1$

- has a unique soln. in $1 + z^{-1}\mathbb{C}[[z^{-1}]]$

Strange derivations and bridge equation of Écalle

(*) non-linear eq$^{\underline{n}}$ $\rightsquigarrow$ $f(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ unique formal solution $\rightsquigarrow$ a family of series $\{f_n(z) \in \mathbb{C}[[z^{-1}]]\}_{n=0}^{\infty}$ s.t. $\sum_{n=0}^{\infty} f_n(z) e^{naz} \sigma^n$ solves (*) $\left(\begin{array}{l} f_0 = f \\ f_1 = 1 + O(z^{-1}) \end{array} \right)$

e.g. (**) $\quad a f(z) - f'(z) = z^{-1} + b z^{-1} f(z)^2$

Plug-in $\sum_{n=0}^{\infty} f_n(z) e^{naz} \sigma^n$ to get :

$$\sum_{n=0}^{\infty} \left(\left((n-1)a + \partial_z \right) \cdot f_n(z) \right) \sigma^n e^{naz} = -z^{-1} - b z^{-1} \left(\sum_{m=0}^{\infty} f_m(z) e^{maz} \sigma^m \right)^2$$

$\rightsquigarrow$ $\quad (-a + \partial_z) f_0(z) = -z^{-1} - b z^{-1} f_0(z)^2$ - original eq$^{\underline{n}}$

$\partial_z f_1(z) = -2b z^{-1} f_0(z) f_1(z)$ $\quad$ linear hgs eq$^{\underline{n}}$ defining $f_1(z) = 1 + O(z^{-1})$.

$(a + \partial_z) f_2(z) = -b z^{-1} \left(f_1(z)^2 + 2 f_0(z) f_2(z) \right)$

$\vdots$ $\quad$ linear inhgs eq$^{\underline{n}}$ defining $f_2(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$

$$\left((n-1)a + \partial_z \right) \cdot f_n(z) = -b z^{-1} \sum_{i+j=n} f_i(z) f_j(z).$$

Thus, if $\varphi_n(t) = \mathcal{B} f_n$, then φ_n has simple singularities at $\{+(n-1)a, (n-2)a, \ldots, 0, -a, -2a, \ldots\}$.

The full information of Stokes' phenomenon is contained in

$$\{\Delta_\ell(f_n)\}_{\substack{\ell \in \mathbb{Z} \\ n \geq 0}} \qquad \Delta_\ell = \text{derivation at } \ell a.$$

So, let

$$F_\ell(z;\sigma) := e^{\ell a z} \cdot \Delta_\ell(F(z;\sigma))$$
$$= \sum_{n=0}^{\infty} \sigma^n e^{(n+\ell)az} \Delta_\ell(f_n(z))$$

Since $e^{\ell a z}\Delta_\ell$ commutes with ∂_z, we get a first order linear equation for $F_\ell(z;\sigma)$. (We are also using the fact that the coefficients of $(*')$ are holomorphic near ∞, and $\Delta_x(\text{hol. fn.}) = 0$.) Now ∂_σ also commutes with ∂_z - and we conclude that $F_\ell(z;\sigma)$ and $\partial_\sigma F(z;\sigma) = \sum_{m=1}^{\infty} m\sigma^{m-1} e^{maz} f_m(z)$ solve the same (linear) equation, hence must be related by some constant (independent of z - but depending on σ)

$$\boxed{F_\ell(z;\sigma) = A_\ell(\sigma)\, \partial_\sigma F(z;\sigma)}$$

e.g. $(**)$ $\partial_z F(z;\sigma) = aF(z;\sigma) - z^{-1} - bz^{-1}F(z;\sigma)^2$

$$\Rightarrow \quad \partial_z F_\ell(z;\sigma) = a F_\ell(z;\sigma) - 2b z^{-1} F(z;\sigma) F_\ell(z;\sigma)$$
$$= \left(a - 2bz^{-1}F(z;\sigma)\right) F_\ell(z;\sigma)$$

Same for $\partial_\sigma F(z;\sigma)$.

From the equation $F_\ell(z;\sigma) = A_\ell(\sigma)\partial_\sigma F(z;\sigma)$ we can conclude that $A_\ell(\sigma) = A_{\ell,-\ell+1}\sigma^{1-\ell}$ (so $A_\ell(\sigma) = 0$ for $\ell \geq 2$)

and

$$\Delta_\ell(f_n) = A_{\ell,-\ell+1}(n+\ell) f_{n+\ell}$$

$\{A_1, A_0, A_{-1}, A_{-2}, \dots\}$ are analytic invariants of $(*)$ and

$$e^{\ell a z} \Delta_\ell(F(z;\sigma)) = A_\ell \sigma^{1-\ell} \frac{\partial}{\partial \sigma} \cdot F(z;\sigma).$$

Hence, for instance

$$\mathcal{S}_0 F(z;\sigma) = F(z;\sigma + A_1)$$
$$\mathcal{S}_\pi F(z;\sigma) = \exp\left(\sum_{m=1}^{\infty} A_{-m}\sigma^{m+1}\partial_\sigma\right) \cdot F(z;\sigma)$$