## §1. Mellin transform.

$$\varphi : \mathbb{R}_{>0} \to \mathbb{C} \rightsquigarrow \tilde{\varphi}(s) := \int_0^\infty t^{s-1} \varphi(t)\, dt$$

(piecewise continuous)

Mellin transform of $\varphi$.

typically holomorphic on a vertical strip $\alpha < \mathrm{Re}(s) < \beta$.

The integral is defined when $\varphi(t) \to 0$ as $t \to \infty$ faster than any polynomial. The domain of $\varphi$, in this case, turns out to be the right-half plane $\mathrm{Re}(s) > -\alpha$, where $\varphi(t) \sim t^\alpha$ as $t \to 0^+$.

<u>Theorem</u> - Assume $\lim_{t \to \infty} t^{-a} \varphi(t) = 0$ for any $a \in \mathbb{R}_{>0}$. Let

$$\varphi(t) \sim \sum_{k=1}^\infty a_k t^{\alpha_k} \text{ as } t \to 0^+ \text{; where } \alpha_1, \dots \in \mathbb{C}$$

and $\alpha = \mathrm{Re}(\alpha_1) \leq \mathrm{Re}(\alpha_2) \leq \dots \to \infty$. Then the integral

$$\int_0^\infty \varphi(t) \cdot t^{s-1}\, dt$$ converges uniformly (rel. to cpct subsets of) $\mathrm{Re}(s) > -\alpha$.

This function extends to a meromorphic $\tilde{\varphi} : \mathbb{C} \dashrightarrow \mathbb{C}$ with simple poles at $-\alpha_1, -\alpha_2, \dots$ of residue $a_1, a_2, \dots$

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Robert Hjalmar Millen 1854-1933

Example. $\varphi(t) = e^{-\lambda t}$ $(\lambda \in \mathbb{R}_{>0})$ $\rightsquigarrow \widetilde{\varphi}(s) = \Gamma(s)\,\lambda^{-s}$

$\varphi(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n t^n$, So $\widetilde{\varphi} : \mathbb{C} \dashrightarrow \mathbb{C}$ is holomorphic on $\{\mathrm{Re} > 0\}$, has simple poles at $0, -1, -2, \ldots$ with residue $\frac{(-1)^n}{n!} \lambda^n$. We already knew this for $\Gamma(s)$.

The proof of the theorem utilizes the following generalization of the Mellin transform:

Let $T \in \mathbb{R}_{>0}$ be arbitrary, and define

$$\widetilde{\varphi}_{\leq T}(s) = \int_0^T t^{s-1} \varphi(t)\, dt = \widetilde{\varphi \cdot \mathbb{1}_{[0,T]}}$$

$$\widetilde{\varphi}_{\geq T}(s) = \int_T^{\infty} t^{s-1} \varphi(t)\, dt = \widetilde{\varphi \cdot \mathbb{1}_{[T,\infty)}}$$

We will prove the following result, which implies the theorem from the previous page, and defines "generalized Mellin transform".

§2. Theorem. Let $\varphi : \mathbb{R}_{>0} \to \mathbb{C}$ be (piecewise) continuous

and let $$\varphi(t) \sim \sum_{k=1}^{\infty} a_k t^{\alpha_k} \quad \text{as } t \to 0^+$$

$$\sim \sum_{\ell=1}^{\infty} b_\ell t^{\beta_\ell} \quad \text{as } t \to \infty.$$

Then $\widetilde{\varphi}_{\leq T}(s)$ is hol. for $\mathrm{Re}(s) > -\alpha$ $\left( \begin{array}{l} \alpha = \mathrm{Re}(\alpha_1) \leq \mathrm{Re}(\alpha_2) \leq \dots \to \infty \\ \beta = \mathrm{Re}(\beta_1) \geq \mathrm{Re}(\beta_2) \geq \dots \to -\infty \end{array} \right)$

and can be extended to $\mathbb{C} \overset{\widetilde{\varphi}_{\leq T}}{\dashrightarrow} \mathbb{C}$

Simple poles at $-\alpha_1, -\alpha_2, \dots$

Residues $a_1, a_2, \dots$

( $\widetilde{\varphi}_{\geq T} : \mathbb{C} \dashrightarrow \mathbb{C}$ hol. on $\mathrm{Re}(s) < -\beta$ ; simple poles at $-\beta_1, -\beta_2, -\beta_3, \dots$ of residues $-b_1, -b_2, \dots$ )

Proof. Let us prove this for lower cut-off function.

The uniform convergence of $\displaystyle\int_0^T \varphi(t) \cdot t^{s-1} dt$ for $\mathrm{Re}(s+\alpha_1) > 0$

is proved exactly as the one for the gamma function. (ACVI, Lecture 17, page 4)

Write $$\widetilde{\varphi}_{\leq T}(s) = \int_0^T \left( \varphi(t) - \sum_{k=1}^{n} a_k t^{\alpha_k} \right) t^{s-1} dt$$

$$+ \int_0^T \sum_{k=1}^{n} a_k t^{s+\alpha_k - 1} dt$$

$$\tilde{\varphi}_{\leq T}(s) = \underbrace{\int_0^T \left(\varphi(t) - \sum_{k=1}^{n} a_k t^{\alpha_k}\right) t^{s-1}\,dt}_{} + \sum_{k=1}^{n} \frac{a_k}{s+\alpha_k} \cdot T^{s+\alpha_k}$$

converges for $\mathrm{Re}(s+\alpha_{n+1}) > 0$

simple poles at $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$

Same argument works for $\tilde{\varphi}_{\geq T}(s)$. □

Define $\boxed{\tilde{\varphi}(s) = \tilde{\varphi}_{\leq T}(s) + \tilde{\varphi}_{\geq T}(s)}$. This is to be viewed as a sum of two mero fns. The effect of changing $T$ to $T'$ is to add $\int_T^{T'} \varphi(t)\, t^{s-1}\,dt$ to $\tilde{\varphi}_{\leq T}(s)$

and subtract it from $\tilde{\varphi}_{\geq T}(s)$.

We call $\tilde{\varphi} : \mathbb{C} \dashrightarrow \mathbb{C}$ with simple singularities at

$-\alpha_1, -\alpha_2, \ldots ; -\beta_1, -\beta_2, \ldots$

w/ residue $a_1, a_2, \ldots ; -b_1, -b_2, \ldots$

§3. Examples.

(1) $\varphi(t) = t^n$ $(n \geq 0)$

$$\tilde{\varphi}_{\leq T}(s) = \int_0^T t^{n+s-1}\,dt = \frac{T^{n+s}}{n+s} \quad \text{for } \mathrm{Re}(s+n) > 0$$

$$\tilde{\varphi}_{\geq T}(s) = \int_T^{\infty} t^{n+s-1}\,dt = -\frac{T^{n+s}}{n+s} \quad \text{for } \mathrm{Re}(s+n) < 0$$

So, $\tilde{\varphi}(s) \equiv 0$.

Exercise - Show that $\tilde{\varphi} \equiv 0$ for $\varphi(t) = t^n \log(t)^m$, $n \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 0}$.

(2) Zeta function and Dirichlet* series. (1837 - Primes in arithmetic progression)

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad ; \quad a_1, a_2, \dots \in \mathbb{C} \quad \leftarrow \text{Dirichlet Series.}$$

By $\boxed{e^{-\lambda t} \mapsto \Gamma(s)\lambda^{-s}, \quad D(s)\cdot\Gamma(s) = \text{Mellin Transform of } \sum_{n=1}^{\infty} a_n e^{-nt}}$

e.g. $a_1 = a_2 = \dots = 1 \rightsquigarrow \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

So, let
$$\varphi(t) = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}$$
$$= \frac{1}{t} - \frac{1}{2} + \sum_{\ell=1}^{\infty} B_{2\ell} \frac{t^{2\ell-1}}{(2\ell)!}$$

Hence $\tilde{\varphi} : \mathbb{C} \dashrightarrow \mathbb{C}$ is hol. on $\mathrm{Re}(s) > 1$

$(\tilde{\varphi} = \Gamma(s)\zeta(s))$ has simple poles at

$1, 0, -1, -2, -3, \dots$

of residue $1 \quad \frac{-1}{2} \quad \frac{B_2}{2!} \quad 0 \quad \frac{B_4}{4!}, \dots$

We know $\Gamma(s)$ has (simple) poles $0, -1, -2, -3, \dots$

of residue $\frac{(-1)^n}{n!}$

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* Johann Peter Gustav Lejeune Dirichlet 1805-1859

Hence $\zeta : \mathbb{C} \dashrightarrow \mathbb{C}$ has simple pole at 1

of residue $\frac{1}{\Gamma(1)} = 1$.

and $$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$ (note : $B_{2k+1} = 0 \ \forall k \geq 1$)

(3) Let $$\vartheta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = 1 + 2\sum_{\ell=1}^{\infty} e^{-\pi \ell^2 t} \quad (t>0)$$

$$\Rightarrow \tilde{\vartheta}(s) = 2\sum_{\ell=1}^{\infty} \Gamma(s)\, \pi^{-s} \ell^{-2s}$$

$$= 2\pi^{-s}\Gamma(s)\zeta(2s) = 2\zeta^*(2s) \text{ where}$$

$$\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

§4. The following result is due to Jacobi

$$\boxed{\vartheta(t) = t^{-1/2}\vartheta(1/t)}$$ - see next page. ~~(left as an exercise)~~

So, $\vartheta(t) \sim t^{-1/2} + O(t^N)$ for any $N>0$

$\Rightarrow \tilde{\vartheta} : \mathbb{C} \dashrightarrow \mathbb{C}$ has simple poles at $\frac{1}{2}$, $0$

of residue $1$, $-1$

Cor. $\zeta^*(s) = \zeta^*(1-s)$

Proof of $\vartheta(t) = t^{-1/2}\,\vartheta\!\left(\frac{1}{t}\right)$ :

Let $\vartheta(t,x) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+x)^2 t}$ : 1-periodic in $x$

So, $\vartheta(t,x) = \sum_{\ell\in\mathbb{Z}} e^{2\pi i \ell x} \cdot c_\ell(t)$ , where $\forall\, \ell\in\mathbb{Z}$,

$$c_\ell(t) = \int_0^1 \vartheta(t,x)\, e^{-2\pi i \ell x}\, dx = \sum_{n\in\mathbb{Z}} \int_0^1 e^{-\pi(n+x)^2 t - 2\pi i \ell x}\, dx$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} e^{-\pi y^2 t - 2\pi i \ell y}\, dy \qquad \begin{array}{c}(y = n+x)\\ e^{2\pi i \ell y} = e^{2\pi i \ell x}\end{array}$$

$$= \int_{-\infty}^{\infty} e^{-\pi t y^2 - 2\pi i \ell y}\, dy = \int_{-\infty}^{\infty} e^{-\pi \frac{\ell^2}{t}}\, e^{-\pi t\left(y + \frac{i\ell}{t}\right)^2} dy$$

$$= e^{-\pi \ell^2/t} \int_{-\infty + i\ell/t}^{+\infty + i\ell/t} e^{-\pi t u^2}\, du \qquad \left(u = y + i\frac{\ell}{t}\right)$$

$$= e^{-\pi \ell^2/t} \int_{-\infty}^{\infty} e^{-\pi t u^2}\, du \qquad \text{(Justify change of contours)}$$

$$= \frac{1}{\sqrt{t}}\, e^{-\pi \ell^2/t}.$$

Hence, $\vartheta(t,x) = \frac{1}{\sqrt{t}} \sum_{\ell \in \mathbb{Z}} e^{-\pi \ell^2/t} \cdot e^{2\pi i \ell x}$, and we get the desired identity by setting $x = 0$.

Now
$$\tilde{\vartheta}(s) = \int_0^\infty \vartheta(t)\, t^{s-1}\, dt$$
$$= \int_0^\infty t^{-1/2}\, \vartheta(t^{-1})\, t^{s-1}\, dt$$
$$= \int_0^\infty \tau^{1/2}\, \vartheta(\tau)\, \tau^{-s+1}\, \tau^{-2}\, d\tau \qquad \left(\tau = \frac{1}{t}\right)$$
$$= \int_0^\infty \vartheta(\tau)\, \tau^{-s+\frac{1}{2}-1}\, d\tau = \tilde{\vartheta}\left(\tfrac{1}{2} - s\right).$$

i.e. $2\zeta^*(2s) = 2\zeta^*(1-2s)$ which proves the functional identity for zeta function. $\square$