Lecture 10

Recall: (Generalized) Mellin Transform of $\varphi(t)$ is $\tilde{\varphi}(s) := \int_0^\infty \varphi(t)\, t^{s-1}\, dt.$

Some elementary transformations:

$$\psi(t) = \varphi(\lambda t) \qquad \tilde{\psi}(s) = \lambda^{-s}\, \tilde{\varphi}(s)$$

$$\psi(t) = t^{\alpha}\varphi(t) \qquad \tilde{\psi}(s) = \tilde{\varphi}(s+\alpha)$$

$$\psi(t) = \varphi(t^{\lambda}) \qquad \tilde{\psi}(s) = \lambda^{-s}\, \tilde{\varphi}(\lambda^{-1} s)$$

$$\psi(t) = \varphi\left(\tfrac{1}{t}\right) \qquad \tilde{\psi}(s) = \tilde{\varphi}(-s)$$

$$\psi(t) = \log(t)\,\varphi(t) \qquad \tilde{\psi}(s) = \tilde{\varphi}'(s)$$

We also proved that

$$\varphi(t) \sim \sum_{n=1}^{\infty} a_n t^{\alpha_n} \text{ as } t \to 0^+ \qquad \Rightarrow \qquad \tilde{\varphi} : \mathbb{C} \dashrightarrow \mathbb{C} \text{ has}$$

$$\sim \sum_{m=1}^{\infty} b_m t^{\beta_m} \text{ as } t \to \infty \qquad \text{simple poles at } -\alpha_1, -\alpha_2, \dots ; -\beta_1, -\beta_2, \dots$$

w/ residue $a_1 \quad a_2, \dots ; -b_1, -b_2, \dots$

Remark - allowing terms of the form $t^{\alpha}(\log t)^m$ changes the order of the pole to $m+1$; - by 5^{th} property listed above

$\varphi(t) \sim t^{c}(\log t)^m \quad \rightsquigarrow \quad \tilde{\varphi}$ has a pole of order $m+1$ at $-c$

$$\frac{(-1)^m \cdot m!}{(s+c)^{m+1}}.$$

We computed a few Mellin transforms last time:

$$\varphi(t) = t^n (\log t)^m \rightsquigarrow \tilde{\varphi} = 0.$$

$$\varphi(t) = e^{-\lambda t} \rightsquigarrow \tilde{\varphi}(s) = \Gamma(s)\,\lambda^{-s}.$$

Hence, for instance, $\varphi(t) = \sum_{n=1}^{\infty} a_n e^{-nt} \rightsquigarrow \tilde{\varphi}(s) = \Gamma(s) \cdot \underbrace{\sum_{n=1}^{\infty} \frac{a_n}{n^s}}_{\text{Dirichlet Series.}}$

§1. Asymptotics of $\sum_{n=1}^{\infty} f(nt)$.

Assume $f : \mathbb{R}_{>0} \to \mathbb{C}$ is a smooth function of sufficiently rapid decay as $t \to \infty$ to ensure the convergence of

$$g(t) = \sum_{n=1}^{\infty} f(nt) \quad ; \text{ e.g. } f(t) = O(t^{-1-\varepsilon}) \text{ as } t \to \infty.$$

Assume that we know asymptotic expansion of f as $t \to 0^+$,

$$f(t) \sim \sum_{n=0}^{\infty} b_n t^n \quad \text{as } t \to 0^+.$$

Theorem. –

$$g(t) \sim \frac{1}{t} \int_0^{\infty} f(x)\,dx + \sum_{n=0}^{\infty} b_n \frac{B_{n+1}}{n+1} (-t)^n$$

as $t \to 0^+$. Here, $\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$ (Bernoulli numbers)

Remark. - (1) We can allow non-integral exponents in the asymptotic expansion of f.

$$f(t) \sim \sum_{Re(\lambda) \geq -1} b_\lambda t^\lambda \quad \text{say} \quad \frac{b_{-1}}{t} + \sum_{n=1}^{\infty} b_n t^{\lambda_n}$$

$$0 \leq Re(\lambda_1) \leq \cdots$$

then
$$\sum_{m=1}^{\infty} f(mt) \sim -\frac{b_{-1}}{t} \log(t) + \frac{1}{t} \int_0^\infty \left(f(x) - \frac{b_{-1} e^{-x}}{x}\right) dx$$

$$+ \sum_{n=1}^{\infty} b_n \zeta(-\lambda_n) (+t)^{\lambda_n}.$$

(2) Allowing for $t^\lambda (\log t)^n$ in the asymptotic expansion of $f(t)$ has the effect of taking n-th derivative w.r.t. λ, of $\zeta(-\lambda) t^\lambda$.

(3) The idea behind this theorem is the following. If $\tilde{f}(s)$ is the Mellin transform of f (similary $\tilde{g}(s)$), then

$$\tilde{g}(s) = \sum_{m=1}^{\infty} m^{-s} \tilde{f}(s) = \zeta(s) \tilde{f}(s)$$

Now $\tilde{f}$ has simple poles at $0, -1, -2, \ldots$
w/ residue $b_0, b_1, b_2, \ldots$

$\Rightarrow$ $\tilde{g}(s)$ has simple poles at $1, 0, -1, -2, \ldots$
of residues $\tilde{f}(1), \; \zeta(-n) b_n = (-1)^n \frac{B_{n+1}}{n+1}$.

So, the asymptotic expansion of $g(t)$ must be of the form

$$\frac{\tilde{f}(1)}{t} + \sum_{n=0}^{\infty} b_n \zeta(-n)\, t^n \,.\quad \text{Note: } \tilde{f}(1) = \int_0^\infty f(t)\,dt.$$

§2. Euler-Maclaurin Sum formula - another proof of Theorem §1.

<u>Definition</u>. - Bernoulli polynomials $B_n(x)$ $(n \geq 0)$ are defined by

$$\frac{t \cdot e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

<u>Properties</u>: (1) $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, x^k$ (in terms of Bernoulli numbers)

$$B_n(0) = B_n$$

(2) $$B_n(x) = \left(\sum_{\ell=0}^{\infty} B_\ell \frac{\partial_x^\ell}{\ell!} \right) \cdot x^n.$$

(3) $$\int_a^b B_n(x)\,dx = \frac{B_{n+1}(b) - B_{n+1}(a)}{n+1} \;;$$

(4) $B_n'(x) = n\, B_{n-1}(x)$ and

$B_n(x+1) - B_n(x) = n\, x^{n-1}$.

Remark.- $B_n(x)$ is the unique poly. s.t. $\displaystyle\int_a^{a+1} B_n(x)\,dx = a^n$.

Euler-Maclaurin Sum formula: $\forall\, M \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{\geq 1}$

$$(*) \quad \int_0^M f(x)\,dx = \sum_{m=1}^{M-1} f(m) + \frac{f(M)+f(0)}{2} + \sum_{n=1}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!}\left(f^{(n)}(M) - f^{(n)}(0)\right)$$

$$+ (-1)^N \int_0^M f^{(N)}(x)\, \frac{\overbrace{B_N(x-[x])}^{\overline{B}_n(x)}}{N!}\,dx$$

(Proof of Thm §1: let $M \to \infty$ in the sum above to get

$$\sum_{m=1}^{\infty} f(m) = \int_0^{\infty} f(x)\,dx + \sum_{n=1}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0)$$

$$- (-1)^N \int_0^{\infty} f^{(N)}(x)\, \frac{\overline{B}_N(x)}{N!}\,dx$$

replacing $f(x)$ by $f(tx)$
and then x by x/t gives:

$$\sum_{m=1}^{\infty} f(mt) = \frac{1}{t}\int_0^{\infty} f(x)\,dx + \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0)\, t^n$$

bounded.

$$+ (-t)^{N-1} \overbrace{\int_0^{\infty} f^{(N)}(x)\, \frac{\overline{B}_N(x/t)}{N!}\,dx}$$ $\square$)

Proof of $(*)$. Integration by parts and the fact that

$$B_1(0) = \frac{-1}{2} = -B_1(1) \quad \text{and} \quad B_{n+1}(1) = B_{n+1}(0) = B_{n+1} \quad \forall\, n \geq 2$$

implies

$$\int_0^1 f^{(n)}(x)\, \frac{B_n(x)}{n!}\, dx = -\int_0^1 f^{(n+1)}(x)\, \frac{B_{n+1}(x)}{(n+1)!}\, dx + \begin{cases} \frac{1}{2}\big(f(0) + f(1)\big) & \text{if } n = 0 \\ \frac{B_{n+1}}{(n+1)!}\big(f^{(n)}(1) - f^{(n)}(0)\big) & \text{if } n \geq 1. \end{cases}$$

By induction, we get

$$\int_0^1 f(x)\, dx = \frac{1}{2}\big(f(0) + f(1)\big) + \sum_{n=1}^{N-1} (-1)^n \frac{B_{n+1}}{(n+1)!}\big(f^{(n)}(1) - f^{(n)}(0)\big) + (-1)^N \int_0^1 f^{(N)}(x)\, \frac{B_N(x)}{N!}\, dx$$

Replace $f(x)$ by $f(x+m-1)$:

$$\int_{m-1}^{m} f(x)\, dx = \frac{1}{2}\big(f(m-1) + f(m)\big) + \sum (-1)^n \frac{B_{n+1}}{(n+1)!}\big(f^{(n)}(m) - f^{(n)}(m-1)\big) + (-1)^N \int_{m-1}^{m} f^{(N)}(x)\, \frac{B_N(x - \overset{m-1}{\overset{\shortparallel}{[x]}})}{N!}\, dx$$

Summing over $m = 1, 2, \ldots, M$ gives $(*)$. $\square$