Lecture 11

Recall - Bernoulli numbers and polynomials are defined as:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad ; \quad \frac{t\, e^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} .$$

§1. Properties of Bernoulli polynomials.

(a) $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, x^k$ since the right-hand side is the coefficient of $\frac{t^n}{n!}$ in $\frac{t}{e^t - 1} \cdot e^{tx}$.

(b) $$\int_a^b B_n(x)\, dx = \frac{B_{n+1}(b) - B_{n+1}(a)}{n+1} .$$

Proof. -

$$\int_a^b \frac{t\, e^{tx}}{e^t - 1}\, dx = \frac{e^{bt} - e^{at}}{e^t - 1} = \frac{1}{t} \sum_{\ell=1}^{\infty} \frac{B_\ell(b) - B_\ell(a)}{\ell!} t^\ell$$

$$= \sum_{n=0}^{\infty} \left(B_{n+1}(b) - B_{n+1}(a)\right) \frac{t^n}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \left(\int_a^b B_n(x)\, dx \right) \frac{t^n}{n!} \qquad \square$$

(c) $$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)\, x^n , \quad \forall\ n \geq 0$$

$$\left(B_0(x+1) = B_0(x) = 1 \right)$$

Proof.- $$\sum_{n=0}^{\infty} B_n(x+1)\frac{t^n}{n!} = \frac{t\,e^{t(x+1)}}{e^t-1}$$

$$= e^{tx}\cdot\frac{t\,e^t}{e^t-1} = e^{tx}\,\frac{t(e^t-1+1)}{e^t-1}$$

$$= t\cdot e^{tx} + \frac{t\cdot e^{tx}}{e^t-1} = \sum_{n=0}^{\infty} x^n\frac{t^{n+1}}{n!} + \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}$$

Comparing coefficients of t^{n+1} $(n\geq 0)$, we get

$$B_{n+1}(x+1) = B_{n+1}(x) + (n+1)x^n \qquad \square$$

(d) Combining (b) and (c) we get

$$\int_a^{a+1} B_n(x)\,dx = a^n \quad \text{for every } n\geq 0.$$

(e) $B_1(1) = \frac{1}{2} = -B_1(0)$ and $B_n(1) = B_n(0) = B_n$ $\forall\, n\neq 1$.

(f) Keeping a fixed and taking derivative w.r.t. b in property (b) above gives

$$B_{n+1}'(x) = (n+1)\,B_n(x).$$

§2. Euler-Maclaurin Summation formula:

$$\int_0^M f(x)\,dx = \sum_{m=1}^{M-1} f(m) + \frac{f(0)+f(M)}{2} + \sum_{n=1}^{N-1} (-1)^n \frac{B_{n+1}}{(n+1)!}\left(f^{(n)}(M) - f^{(n)}(0)\right)$$
$$+ (-1)^N \int_0^M f^{(N)}(x)\,\frac{B_N(x-[x])}{N!}\,dx.$$

Proof.-
$$\int_0^1 f^{(n)}(x)\,\frac{B_n(x)}{n!}\,dx = -\int_0^1 f^{(n+1)}(x)\,\frac{B_{n+1}(x)}{(n+1)!}\,dx + \left[f^{(n)}(x)\cdot\frac{B_{n+1}(x)}{(n+1)!}\right]_{x=0}^{x=1}$$

$$\Rightarrow \int_0^1 f^{(n)}(x)\,\frac{B_n(x)}{n!}\,dx = -\int_0^1 f^{(n+1)}(x)\,\frac{B_{n+1}(x)}{(n+1)!} + \begin{cases} \frac{1}{2}\left(f(0)+f(1)\right) & \text{if } n=0 \\ \frac{B_{n+1}}{(n+1)!}\left(f^{(n)}(1) - f^{(n)}(0)\right) & \text{if } n \geq 1 \end{cases}$$

By an easy induction argument, we get

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left(f(0)+f(1)\right) + \sum_{n=1}^{N-1} (-1)^{n\cancel{+1}} \frac{B_{n+1}}{(n+1)!}\left(f^{(n)}(1) - f^{(n)}(0)\right)$$
$$+ (-1)^N \int_0^1 f^{(N)}(x)\,\frac{B_N(x)}{N!}\,dx$$

Replace $f(x)$ by $f(x+m-1)$ to get

$$\int_{m-1}^{m} f(x)\,dx = \frac{1}{2}\Big(f(m-1)+f(m)\Big) + \sum_{n=1}^{N-1} (-1)^n \frac{B_{n+1}}{(n+1)!}\Big(f^{(n)}(m) - f^{(n)}(m-1)\Big)$$

$$+ (-1)^N \underbrace{\int_0^1 f^{(N)}(x+m-1)\,\frac{B_N(x)}{N!}\,dx}$$

$$\hookrightarrow \int_{m-1}^{m} f^{(N)}(x)\,\frac{B_N(x-m+1)}{N!}\,dx$$

and we get the claimed formula by adding these equations $m = 1, 2, \ldots, M-1, M$. $\square$

§3. Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function, $f^{(n)}(t) \to 0$ as $t \to \infty$ for every n

and $f(t) \sim \sum_{n=0}^{\infty} b_n t^n$ as $t \to 0^+$.

Let $g(t) = \sum_{m=1}^{\infty} f(mt)$. Then,

$$g(t) \sim \frac{1}{t}\int_0^{\infty} f(x)\,dx + \sum_{n=0}^{\infty} b_n\, \zeta(-n)\, t^n \qquad \left(\text{recall : } \zeta(-n) = \frac{(-1)^n B_{n+1}}{n+1}\right)$$

Proof.- let $M \to \infty$ in Euler-Maclaurin formula to get

$$\sum_{m=1}^{\infty} f(m) = \int_0^{\infty} f(x)\,dx + \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0)$$
$$- (-1)^N \int_0^{\infty} f^{(N)}(x) \frac{B_N(x-[x])}{N!}\,dx.$$

Replace f by $f(tx)$ and perform the change of variables $y = tx$ to get

$$\sum_{m=1}^{\infty} f(mt) = \frac{1}{t}\int_0^{\infty} f(y)\,dy + \sum_{n=0}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0)\, t^n$$
$$+ (-t)^{N-1} \int_0^{\infty} f^{(N)}(y) \frac{\overline{B}_N(y/t)}{N!}\,dy.$$

and the result follows $\square$

§4. Thus it follows that $f \rightsquigarrow g(t) = \sum_{m=1}^{\infty} f(mt)$ has the effect of

Changing t^n to $\zeta(-n)\,t^n$ and introducing a singular term

$\frac{1}{t}\int_0^{\infty} f(x)\,dx$ - if the latter integral makes sense.

This result can be extended to include non-integral exponents of t

For t^{-1} term, we can use $\sum_{m=1}^{\infty} \frac{e^{-mt}}{mt} = -\frac{\log(1-e^{-t})}{t}$ to get

$$f(t) \sim \frac{b_{-1}}{t} + \sum_{n=0}^{\infty} b_n t^n \quad \Rightarrow \quad g(t) \sim -\frac{b_{-1}\log(t)}{t} + \frac{1}{t}\int_0^{\infty}\left(f(t) - b_{-1}\frac{e^{-t}}{t}\right) dt$$

as $t \to 0$

$$+ \cdots$$

Allowing terms of the form $\log(t)^n \cdot t^c$ changes $\zeta(-c)t^c$ to $\partial_\lambda^n \cdot \left(\zeta(-\lambda)\cdot t^\lambda\right)\Big|_{\lambda=c}$. Thus, for instance

$$f(t) \sim -b\log(t) + \sum_{n=0}^{\infty} b_n t^n \Rightarrow g(t) \sim \frac{b}{2}\log\left(\frac{t}{2\pi}\right) + \frac{1}{t}\cdot I + \sum_{n=0}^{\infty} b_n \zeta(-n) t^n$$

using

$$\partial_\lambda\left(\zeta(-\lambda)t^\lambda\right) = -\zeta'(-\lambda)t^\lambda + \zeta(-\lambda)\log(t)\cdot t^\lambda$$

$$\underset{(\text{at } \lambda=0)}{=} -\zeta'(0) + \zeta(0)\log(t) = \frac{1}{2}\log(2\pi) - \frac{1}{2}\log(t).$$

§5. Example $\quad P(x) = \prod_{m=1}^{\infty} \frac{1}{1-x^m}$ generating series for the partition function.

$$\log P(x) = \sum_{m=1}^{\infty}\left(-\log(1-x^m)\right)$$

So $\log P(e^{-t}) = \sum_{m=1}^{\infty} f(mt)$ where $f(t) = -\log(1-e^{-t})$

$$f(t) \sim -\log(t) - \sum_{n=1}^{\infty} \frac{B_n}{n\cdot n!} t^n \quad \text{as } t \to 0$$

$$\Rightarrow \log P(e^{-t}) \sim \frac{I}{t} + \frac{1}{2}\log\left(\frac{t}{2\pi}\right) - \underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{B_n B_{n+1}}{n\cdot(n+1)!} t^n}_{\text{all terms except } n=1 \text{ vanish}}$$

$$I = \int_0^\infty f(t)\,dt = \sum_{m=1}^{\infty}\int_0^\infty \frac{e^{-mt}}{m}\,dt = \zeta(2) = \frac{\pi^2}{6}$$

i.e., $\log(P(e^{-t})) \sim \dfrac{\pi^2}{6t} + \dfrac{1}{2}\log\left(\dfrac{t}{2\pi}\right) - \dfrac{1}{24}t + O(t^N)$ for any $N>0$.

This is usually written as

$$\log P(e^{-2\pi t}) = \frac{\pi}{12t} + \frac{1}{2}\log(t) - \frac{\pi}{12}t + O(t^N) \quad \text{for any } N>0$$

and can be obtained from the following modularity property of P :

$$e^{\frac{\pi}{12}t} P(e^{-2\pi t}) = t^{1/2} e^{\frac{\pi}{12t}} P(e^{-2\pi/t}). \quad \text{(for later...)}$$

§6. Example. Let $\sigma_\ell(n) = \sum_{d|n} d^\ell$ (here $\ell \geq 0$)

For $k \geq 1$, let $g_k(x) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) \cdot x^n$.

Then $g_k(x) = \sum_{m=1}^{\infty} m^{k-1} \dfrac{x^m}{1-x^m}$ (Ex. Verify this.)

So, $g_k(e^{-t}) = \dfrac{1}{t^{k-1}} \sum_{m=1}^{\infty} f_k(mt)$ where $f(t) = \dfrac{t^{k-1}}{e^t - 1}$

$\int_0^\infty f(x)\,dx = (k-1)!\,\zeta(k)$ and $f(t) = \sum_{r=0}^{\infty} \dfrac{B_r}{r!} t^{r+k-2}$ near $t=0$

$$\Rightarrow g_k(e^{-t}) \sim \frac{(k-1)!\,\zeta(k)}{t} + \sum_{r=0}^{\infty} (-1)^{r+k} \frac{B_r\, B_{r+k-1}}{r!\; r+k-1} t^{r-1}$$