Lecture 12

§1. A duplication formula for Gamma function. (Gauss, Legendre)

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}\; n^{\frac{1}{2}-nz}\;\Gamma(nz)$$

Proof.- Let $\varphi(z) = \dfrac{n^{nz}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)}{n\cdot\Gamma(nz)}$

By Euler's formula $\Gamma(x) = \lim\limits_{m\to\infty} \dfrac{(m-1)!\; m^{x}}{x(x+1)\cdots(x+m-1)}$

$$\varphi(z) = \frac{n^{nz}\displaystyle\prod_{r=0}^{n-1}\lim_{m\to\infty}\frac{(m-1)!\;m^{z+\frac{r}{n}}}{\left(z+\frac{r}{n}\right)\cdots\left(z+\frac{r}{n}+m-1\right)}}{n\displaystyle\lim_{m\to\infty}\frac{(m-1)!\;m^{nz}}{nz\,(nz+1)\cdots(nz+m-1)}}$$

Change m in the denominator to nm

$$\varphi(z) = \frac{n^{nz}\displaystyle\prod_{r=0}^{n-1}\lim_{m\to\infty}\frac{(m-1)!\;m^{z+\frac{r}{n}}}{\left(z+\frac{r}{n}\right)\cdots\left(z+\frac{r}{n}+m-1\right)}}{n\displaystyle\lim_{m\to\infty}\frac{(nm-1)!\;(nm)^{nz}}{nz\,(nz+1)\cdots(nz+nm-1)}}$$

$$= n^{\cancel{nz}-1}\lim_{m\to\infty}\frac{((m-1)!)^n\; m^{\cancel{nz}+\frac{n-1}{2}}\; n^{mn}}{(nm-1)!\;\cancel{(nm)^{nz}}} \quad \text{independent of } z.$$

$$\varphi\left(\frac{1}{n}\right) = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$$

$$\left(\varphi\left(\frac{1}{n}\right)\right)^2 = \prod_{r=1}^{n-1}\Gamma\left(\frac{r}{n}\right)\Gamma\left(1-\frac{r}{n}\right)$$ use $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

$$= \frac{\pi^{n-1}}{\sin\left(\frac{\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)}$$

Ex. $\sin\left(\frac{\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$.

$$= \frac{(2\pi)^{n-1}}{n}$$ and we are done. □

For $n=2$, $2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z)$ is due to Legendre.

§2. <u>Functional equation for $\zeta(s)$</u>. – Recall, in Lecture 9 (p.6)

we showed

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{-1+s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

Using duplication formula and sine identity, we get:

$$\frac{\zeta(1-s)}{\zeta(s)} = \pi^{\frac{1}{2}}\,\pi^{-s}\,\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\cdot\frac{\Gamma\left(1-\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{1-s}{2}\right)}$$

$$= \pi^{1/2}\,\pi^{-s}\,\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\pi}\cdot\underbrace{\sin\left(\pi\left(\frac{1-s}{2}\right)\right)}_{\hookrightarrow\ \cos\left(\frac{\pi s}{2}\right)}$$

$$= \frac{\cancel{\pi}^{1/2}\,\pi^{-s}}{\cancel{\pi}}\;\cancel{\pi}^{1/2}\,\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\cdot 2^{1-s}$$

$$\boxed{\zeta(1-s) = 2^{1-s}\,\pi^{-s}\,\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)}$$

The values of $\zeta(0)$, $\zeta'(0)$ etc. can be computed from this identity. e.g. multiply by s and let $s \to 0$ to get

$$-\operatorname{Res}_1 \zeta = 2\,\zeta(0) \implies \zeta(0) = \frac{-1}{2}.$$

and ζ has residue 1 at 1

Ex. $\zeta'(0) = \frac{-1}{2}\log(2\pi)$.

§3. Asymptotics of sums of the form $\sum_{n=1}^{\infty} f(nt)$.

Recall from the previous lecture, Euler-Maclaurin formula

$$\sum_{m=1}^{M-1} f(m) = \int_0^M f(x)\,dx + \frac{f(M)+f(0)}{2} - \sum_{n=1}^{N-1} \frac{(-1)^n B_{n+1}}{n+1}\left(f^{(n)}(M) - f^{(n)}(0)\right)$$
$$- (-1)^n \int_0^M f^{(N)}(x)\,\frac{B_N(x-[x])}{N!}\,dx$$

Using this, we showed the following special case

$$f(t) \sim -b \log(t) + \sum_{n=0}^{\infty} b_n t^n \quad (\text{as } t \to 0^+)$$

$$\Rightarrow \quad \sum_{m=1}^{\infty} f(mt) \sim \frac{1}{t} \int_0^{\infty} f(x)\,dx + \frac{b}{2} \log\left(\frac{t}{2\pi}\right) + \sum_{n=0}^{\infty} b_n \zeta(-n) t^n .$$

§4. Side remark on Ramanujan sums.

To a series of the form $\sum_{m=1}^{\infty} f(m)$, Ramanujan associates a constant

$$C_f := -\int_0^1 f(x)\,dx + \frac{f(1)}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(1)$$

Using Euler-Maclaurin sum formula, we can show that

$$\sum_{m=1}^{\infty} f(m) = C_f + \int_0^{\infty} f(x)\,dx$$

In Hardy's notation (Divergent Series, Chapter 13) a more general definition is given, depending on N and $0 \leq a < 1$ for $N \to \infty$ and $a = 0$.

So, our C_f is Hardy's one, for $N \to \infty$ and $a = 0$.

Let us call Ramanujan sum $\sum_{n=1}^{\infty} f(n) = C_f \quad (R).$

e.g. $f(n) = 1 \quad \forall n \; ; \quad C_f = -1 + \frac{1}{2} = -\frac{1}{2} = \zeta(0).$

$$f(n) = n \quad \forall n \; ; \quad C_f = -\int_0^1 x\,dx + \frac{1}{2} - \frac{B_2}{2}$$

$$= \frac{-1}{12}$$

Remark.- Ramanujan sum does not satisfy translation property

$$\sum_{n=1}^{\infty} {}^{R} f(n) \neq f(1) + \sum_{n=2}^{\infty} {}^{R} f(n).$$

(e.g. take $f(n) = 1$ example).

§5. Partition function example

$$P(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$$

$$\log P(e^{-t}) = \sum_{n=1}^{\infty} -\log(1 - e^{-tn}) = \sum_{n=1}^{\infty} f(nt)$$

$$f(t) = -\log(1 - e^{-t}). \qquad \int_0^{\infty} f(x)\,dx = \zeta(2) = \frac{\pi^2}{6}$$

$$f(t) = -\log t - \sum_{n=1}^{\infty} \frac{B_n}{n \cdot n!} t^n$$

$$\Rightarrow \log(P(e^{-t})) \sim \frac{\zeta(2)}{t} + \frac{1}{2}\log\left(\frac{t}{2\pi}\right) - \underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{B_n B_{n+1}}{n\cdot(n+1)!} t^n}_{\text{only } n=1 \text{ term contributes}}$$

$$\sim \frac{\pi^2}{6t} + \frac{1}{2}\log\left(\frac{t}{2\pi}\right) - \frac{t}{24}$$

$$+ O(t^N) \quad \forall N > 0.$$

$$\frac{B_1 B_2}{2} = \frac{-1}{24}$$

$$\log(P(e^{-2\pi t})) \sim \frac{\pi}{12t} - \frac{\pi t}{12} + \frac{1}{2}\log(t) \quad \text{as } t \to 0.$$

<u>Ex.</u> More generally, let $h, k \in \mathbb{Z}_{\geq 1}$; $\gcd(h,k) = 1$; $h \leq k$.

$\cancel{\log P(1 - e}$

$$\log P\left(e^{2\pi i \frac{h}{k}} e^{-2\pi \frac{t}{k^2}}\right) \sim \frac{\pi}{12t} - \frac{\pi t}{12k^2} + \frac{1}{2}\log\left(\frac{t}{k}\right)$$

$$+ \pi i \underbrace{s(h,k)}_{\text{called Dedekind sum.}}$$

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)$$

Recall - Modularity of Dedekind's η-function :

$$P(e^{-2\pi t}) = \sqrt{t}\ e^{\left(\frac{\pi}{12t} - \frac{\pi t}{12}\right)}\ P(e^{-2\pi/t})\ (t>0)$$

can also be used to compute these asymptotics as $t \to 0$.

i.e. $x = e^{-2\pi t} \to 1^-$.

More general modularity relation for $P(x)$ is known in terms of the Dedekind sums $s(h,k)$

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \qquad Im(\tau) > 0$$

$$\eta(-1/\tau) = (-i\tau)^{1/2}\ \eta(\tau)$$