Lecture 13

§1. Hardy-Ramanujan ; Rademacher formula for $p(n)$.

Recall $p(n) =$ number of partitions of n

$$= \left| \left\{ \underset{\lambda_j \in \mathbb{Z}_{\geq 0}}{(\lambda_1 \geq \lambda_2 \geq \dots)} \;\middle|\; \sum_{j\geq 1} \lambda_j = n \right\} \right|$$

$$= \left| \left\{ \underset{r_j \in \mathbb{Z}_{\geq 0}}{(r_1, r_2, r_3, \dots)} \;\middle|\; \sum_{j \geq 1} j \cdot r_j = n \right\} \right|$$

$$1 + \sum_{n=1}^{\infty} p(n) x^n \underset{\text{(Euler)}}{=} \prod_{m=1}^{\infty} \frac{1}{1-x^m} =: P(x)$$

Last time we showed

$$P\left(e^{2\pi i \frac{h}{k} - \frac{2\pi t}{k^2}}\right) \underset{\text{as } t \to 0}{\sim} \omega(h,k) \sqrt{\frac{t}{k}} \exp\left(\frac{\pi}{12t} - \frac{\pi t}{12k^2}\right).$$

$$(1 + O(t^N)) \quad \forall N > 0.$$

$$\omega(h,k) = e^{\pi i \underbrace{s(h,k)}_{\text{Dedekind sum.}}}$$

Theorem. -
$$\boxed{p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \, \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right)}$$

where
$$A_k(n) = \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \omega(h,k) \cdot e^{-2\pi i n \frac{h}{k}}$$

§2. Idea of the proof – Farey fractions and Ford circles.

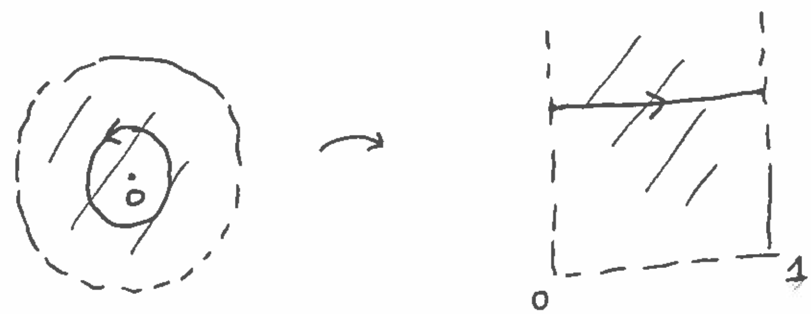
($n \geq 1$ is fixed in this section)

Start from Cauchy's formula

$$p(n) = \frac{1}{2\pi i} \int_C \frac{P(x)}{x^{n+1}}\, dx$$

C : circle of radius < 1 centered at 0 (positively oriented)

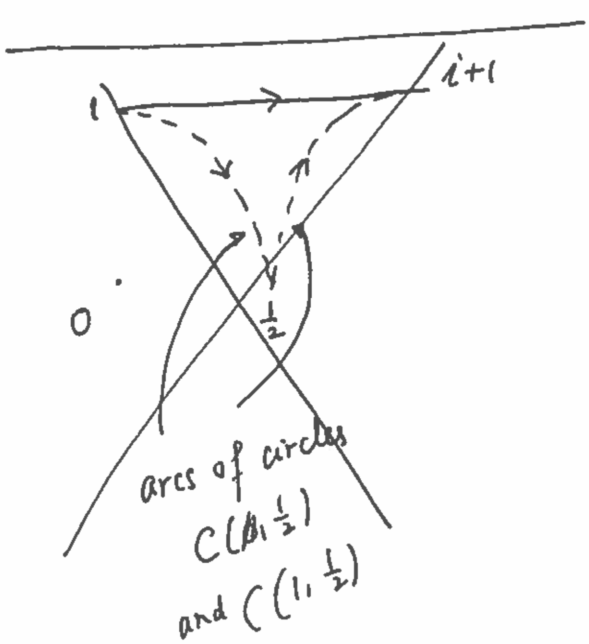
Set $x = e^{2\pi i \tau}$

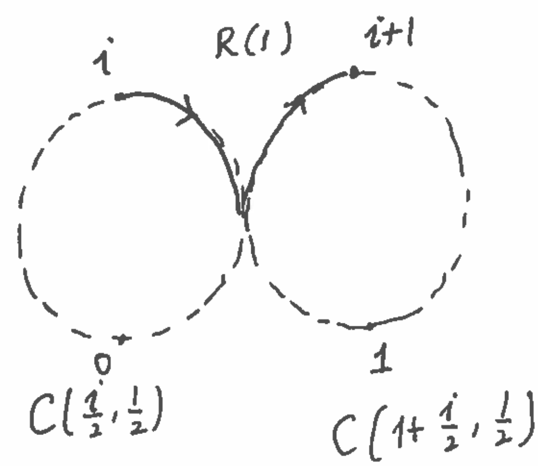


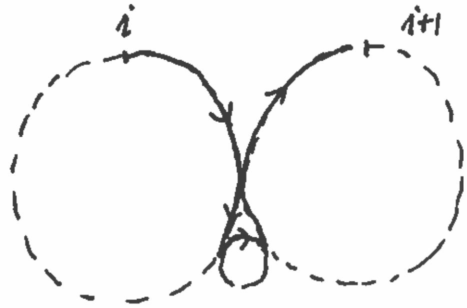
$|x| < 1$ $\qquad$ $Im(\tau) > 0$

Take $C = C(0, e^{-2\pi})$ which maps to the segment $i \to 1+i$

$\to$ We replace this line segment by "Rademacher path $R(N)$" consisting of upper arcs of "Ford circles to order N".







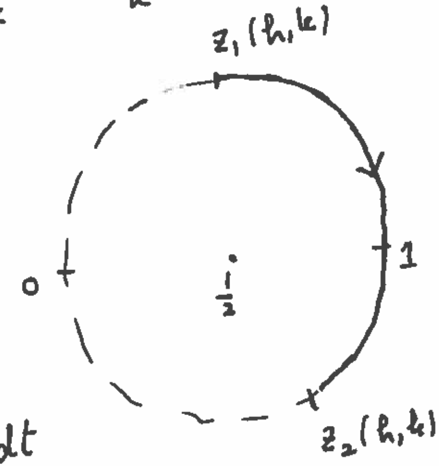
and so on ...

R(2)

Remark.- Next part of this lecture is devoted to the properties of Ford circles and Farey fractions. For now, let us remark that R(N) consists of arcs $\gamma_{h,k}$: $\begin{array}{l} 1 \le k \le N \\ 0 \le h < k \\ \gcd(h,k)=1 \end{array}$

So,
$$p(n) = \int_i^{i+1} P(e^{2\pi i \tau})\, e^{-2\pi i n \tau}\, d\tau$$
$$= \sum_{h,k} \int_{\gamma_{h,k}} P(e^{2\pi i \tau})\, e^{-2\pi i n \tau}\, d\tau$$

Yet another change of variables $\tau = \frac{h}{k} + \frac{i\,t}{k^2}$ maps $\gamma_{(h,k)}$ to arc of $C(\frac{1}{2}, \frac{1}{2})$ and each integral (h,k) changes to



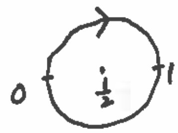
$$i\, k^{-2}\, e^{-2\pi i n h/k} \int_{z_1(h,k) \to z_2(h,k)} e^{\frac{2\pi n t}{k^2}}\, P\left(e^{\frac{2\pi i h}{k} - \frac{2\pi t}{k^2}}\right) dt$$

We then show that replacing P by the r.h.s. of the asymptotic formula $\omega(h,k)\, t^{+1/2} k^{-1/2} \exp\left(\frac{\pi}{12t} - \frac{\pi t}{12k^2}\right)$

introduces $O(N^{-1/2})$ error which is insignificant as $N \to \infty$.

Similarly replacing the arc $z_1 \to z_2$ by the entire circle introduces only 0 as $N \to \infty$ error.

$$p(n) = i \sum_{K=1}^{\infty} A_K(n)\, K^{-5/2} \int t^{1/2}\, e^{\frac{\pi}{12t} + \frac{2\pi t}{K^2}\left(n - \frac{1}{24}\right)}\, dt$$



Möbius transformation $t \mapsto \frac{1}{w}$ changes this circle to line $1 + i\mathbb{R}$

$$p(n) = \frac{1}{i} \sum_{K=1}^{\infty} A_K(n)\, K^{-5/2} \underbrace{\int_{1-i\infty}^{1+i\infty} w^{-5/2} \exp\left(\frac{\pi w}{12} + \frac{2\pi}{K^2}\left(n - \frac{1}{24}\right)\frac{1}{w}\right) dw}_{\text{Bessel integral (for later)}}$$

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}}\, \frac{d}{dz}\left(\frac{\sinh(z)}{z}\right)$$

$$\frac{(z/2)^{3/2}}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-3/2-1} \exp\left(y + \frac{z^2}{4y}\right) dy$$

§3. Farey fractions of order N are all rational numbers from $[0,1]$ whose reduced expression $\frac{a}{b}$ has denominator $b \leq N$.
(i.e. $\gcd(a,b)=1$)

These are ordered in increasing magnitude.

$$F_1: \quad \frac{0}{1} \quad \frac{1}{1}$$

$$F_2: \quad \frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1}$$

$$F_3: \quad \frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1}$$

$$F_4: \quad \frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}$$

$$F_5: \quad \frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{1}{1}$$

...

Properties of Farey fractions are derived from the following easy fact:

If $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc-ad}{b(b+d)} > 0 \quad \text{since } bc-ad > 0.$$

$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc-ad}{d(b+d)} > 0$$

(i) If $\frac{a}{b} < \frac{c}{d}$ and $bc-ad=1$, then $\frac{a}{b}, \frac{c}{d}$ are consecutive in F_N for $\max(b,d) \leq N \leq b+d-1$.

Proof.- Note that $bc-ad=1 \Rightarrow \gcd(a,b) = 1 = \gcd(c,d)$.

$$\max(b,d) \leq N \iff \frac{a}{b}, \frac{c}{d} \in F_N.$$

Now, if $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$ then

$$\begin{aligned} k &= (bc-ad)\,k \\ &= b(ck-hd) + d(bh-ak) \\ &\geq b+d \qquad \square \end{aligned}$$

(ii) $\frac{a}{b} < \frac{c}{d}$, $bc-ad=1$. Let $\frac{h}{k} = \frac{a+c}{b+d}$ $\cancel{(\gcd(h,k)=1)}$.

Then $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$ and $bh-ka = 1 = ck-dh$.

(in the proof above $k = b+d \iff ck-hd = bh-ak = 1$.)

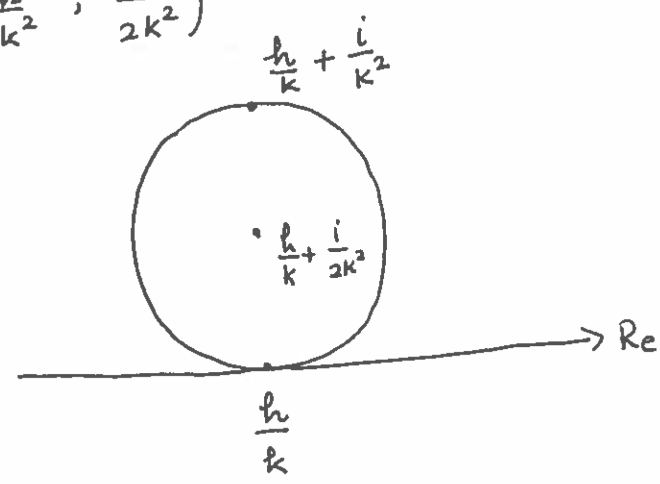
(iii) F_{N+1} includes F_N. Each fraction in $F_{N+1} \setminus F_N$ is of the form $\frac{a+c}{b+d}$ for $\frac{a}{b} < \frac{c}{d}$ consecutive in F_N.

Moreover $\frac{a}{b} < \frac{c}{d}$ are consecutive in any $F_N \iff bc-ad=1$.

(easy induction argument based on (i) & (ii) above.)

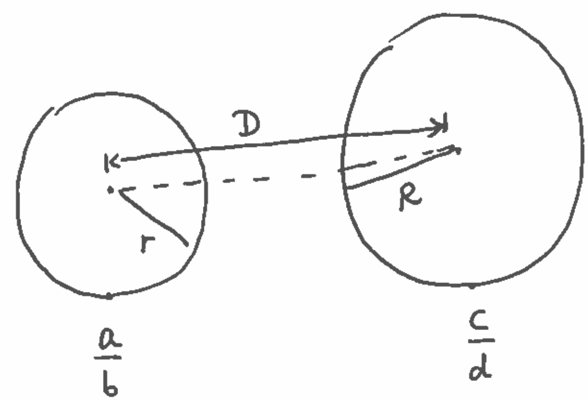
§4. Ford Circles. - Given $\frac{h}{k}$ with $\gcd(h,k) = 1$,

$$C(h,k) = C\left(\frac{h}{k} + i\frac{1}{2k^2}, \frac{1}{2k^2}\right)$$



Prop. - $C(a,b)$ and $C(c,d)$ are either tangent to each other or do not intersect.

They are tangent $\iff |bc - ad| = 1$.



$$D^2 = \left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2$$

$$(r+R)^2 = \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2$$

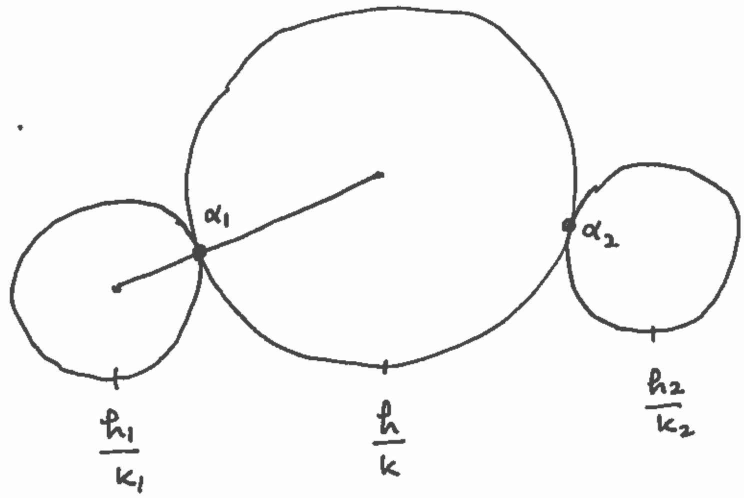
$$\Rightarrow D^2 - (r+R)^2 = \frac{(ad-bc)^2 - 1}{b^2 d^2} \geq 0.$$

It is equal to 0 iff $|ad - bc| = 1$ $\square$

§5. Let $\cancel{\frac{h}{k} < \frac{h_1}{k_1} < h}$ $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ be three consecutive Farey fractions.

Let α_1 (α_2) be the point of tangency

between $C(h,k)$ & $C(h_1,k_1)$ $(C(h_2,k_2))$.

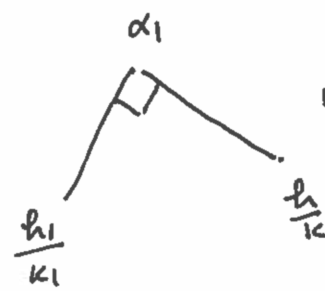


Then,

$$\alpha_1 = \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + \frac{i}{k^2+k_1^2}$$

$$\alpha_2 = \frac{h}{k} + \frac{k_2}{k(k^2+k_2^2)} + \frac{i}{k^2+k_2^2}.$$

Moreover, the angle suspended at α_1 by



is $\frac{\pi}{2}$

Hint :

$$\frac{a}{\frac{h}{k} - \frac{h_1}{k_1}} = \frac{\frac{1}{2k_1^2}}{\frac{1}{2k^2} + \frac{1}{2k_1^2}}$$

$$\frac{b}{\frac{1}{2k^2}} = \frac{\frac{1}{2k^2} - \frac{1}{2k_1^2}}{\frac{1}{2k^2} + \frac{1}{2k_1^2}}$$