Lecture 14

§1. Hypergeometric series

$$F(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)} \frac{z^n}{n!}$$

(Euler, Gauss, Riemann, Barnes, Mellin ...)

Some special cases : (i) if $b=c$, then $F(a,b;b;z) = \sum_{n=0}^{\infty} \binom{a+n-1}{n} z^n$

$$= (1-z)^{-a}.$$

(ii) $\log(1+z) = z\, F(1,1;2;-z)$

We will assume that $c \notin \mathbb{Z}_{\leq 0}$. If a or $b \in \mathbb{Z}_{\leq 0}$, then $F(a,b;c;z)$ is a polynomial and hence has infinite radius of convergence.

Otherwise F has radius of convergence $= 1$.

Ex. $\frac{d}{dz} F(a,b;c;z) = \frac{ab}{c} F(a+1,b+1;c+1;z)$

§2. Differential equation :

$F(a,b;c;z)$ solves the following 2^{nd} order linear ODE, called the hypergeometric equation

$$\boxed{z(1-z)\frac{d^2 f}{dz^2} + (c-(a+b+1)z)\frac{df}{dz} - ab f = 0}$$

In terms of $\delta_z = z \frac{d}{dz}$, this equation takes the following form:

$$\left\{ \delta_z \left(\delta_z + c - 1 \right) - z \left(\delta_z + a \right) \left(\delta_z + b \right) \right\} \cdot f = 0 .$$

Hypergeometric differential equation has 3 (regular) singular points: 0, 1 and ∞. Its exponents are

0 : 0 and $1-c$

∞ : a and b

1 : 0 and $c-a-b$

Exponent of a linear ODE near a singular point z_0 is α such that we have a formal solution of the form $(z-z_0)^{\alpha} \left(1 + \sum_{n=1}^{\infty} c_n (z-z_0)^n \right)$

e.g. Compute the formal solution of the hypergeometric equation which is of the form $z^{1-c} \left(1 + \cdots \right)$.

Remark. Kummer (1836) obtained 24 identities among hypergeometric functions in z, $1-z$ and z^{-1} variables. reflecting the effect of a change of variables preserving the set $\{0, 1, \infty\}$.

§3. Euler's formula.

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt$$

if $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$.

Proof.- $$F(a,b;c;x) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(a)\ \Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+m)} \frac{x^m}{m!}$$

recall $$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$ if $\operatorname{Re}(x)>0$, $\operatorname{Re}(y)>0$

So, $$F(a,b;c;x) = \frac{1}{B(b,c-b)} \sum_{m=0}^{\infty} B(b+m,\, c-b) \frac{\Gamma(a+m)}{\Gamma(a)\cdot m!} x^m$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \sum_{m=0}^{\infty} \int_0^1 t^{b-1}(1-t)^{c-b-1} \binom{a+m-1}{m} \frac{t^m x^m}{m!}\,dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt$$

$\square$

§4. Barnes' integral formula - idea:

if
$$f(x) = \frac{1}{2\pi i} \int_C g(t)\,(-x)^t\, dt$$
C unspecified for now.

solves the hypergeometric equation.

then
$$(x\,\partial_x)\cdot f = \frac{1}{2\pi i}\int_C t\cdot g(t)\,(-x)^t\,dt$$
$$(-x)\cdot f = \frac{1}{2\pi i}\int_C g(t)\,(-x)^{t+1}\,dt$$
$$= \frac{1}{2\pi i}\int_{C+1} g(t-1)\,(-x)^t\,dt.$$

turns the differential equation for $f(z)$ into a "difference equation"

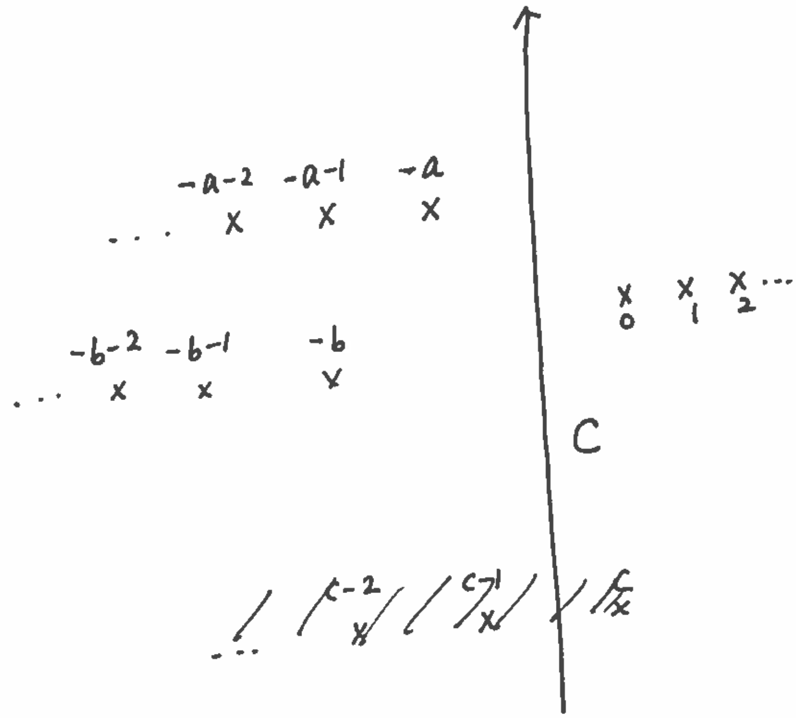
for $g(t)$
$$\int_{\mathcal{C}} t\,(t+c-1)\,g(t)\,(-x)^t\,dt + \int_{\mathcal{C}+1}(s+a-1)(s+b-1)\,g(s-1)\,(-x)^s\,ds = 0$$

Assuming deforming $\mathcal{C}+1$ to $\mathcal{C}$ can be justified, we get
$$\boxed{g(t+1) = -\frac{(t+a)(t+b)}{(t+c)\,(t+1)}\,g(t)}$$

Theorem.- (Barnes)

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a,b;c;z) = \frac{1}{2\pi i}\int_C \frac{\Gamma(t+a)\Gamma(t+b)}{\Gamma(t+c)}\,\Gamma(-t)\,(-x)^t\,dt$$

C is an infinite contour from $-i\infty$ to $+i\infty$, indented so as to have $\{a, a-1, \dots, b, b-1, \dots\}$ to its left and $0, 1, 2\dots$ to its right (assume $a, b \notin \mathbb{Z}_{\leq 0}$).

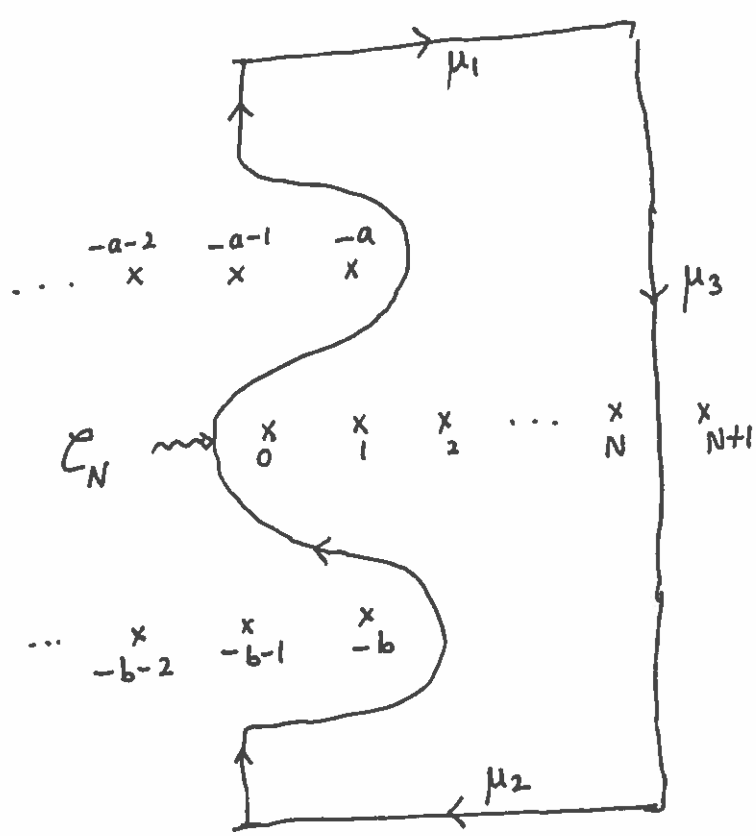


Singularities of the integrand

Sketch of a proof.-

$$\frac{1}{2\pi i}\int_{\mathcal{C}_N+\mu_1+\mu_2+\mu_3} \frac{\Gamma(t+a)\Gamma(t+b)}{\Gamma(t+c)}\,\Gamma(-t)(-x)^t\,dt$$

$$= \sum_{k=0} \frac{\Gamma(k+a)\Gamma(k+b)}{\Gamma(k+c)}\,\frac{x^k}{k!}$$



$\square$

Cor. $$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)} F(a,b;c;z) = \frac{\Gamma(a)\,\Gamma(b-a)}{\Gamma(c-a)}(-z)^{-a} F(a;1-c+a;1-b+a;z^{-1})$$

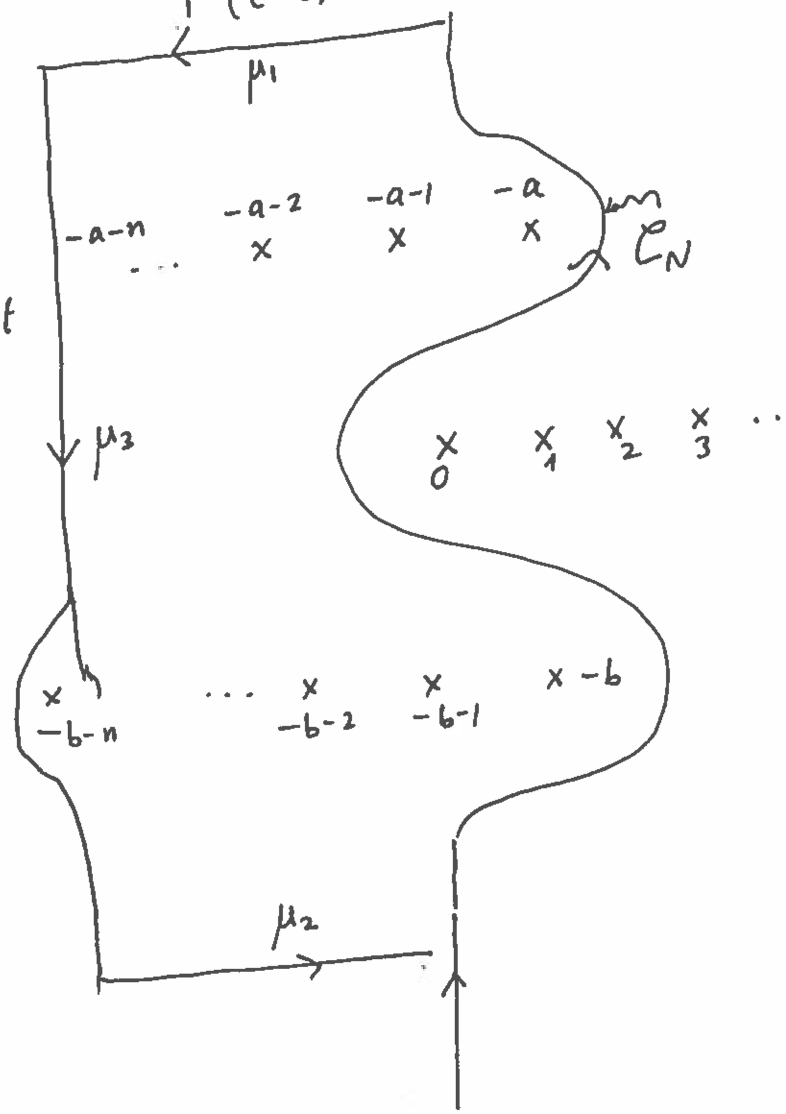
$$+ \frac{\Gamma(b)\,\Gamma(a-b)}{\Gamma(c-b)}(-z)^{-b} F(b;1-c+b;1-a+b;z^{-1})$$

Sketch of a proof.-

$$\frac{1}{2\pi i}\int_{\mathcal{C}_N+\mu_1+\mu_2+\mu_3} \frac{\Gamma(t+a)\,\Gamma(t+b)}{\Gamma(t+c)}\Gamma(-t)(-x)^t\,dt$$

$$= \sum_k \text{residues at } -a-k$$

$$+ \sum_k \text{residue at } -b-k$$



Residue of the integrand at $-a-n$: $$\Gamma(a+n)\,(-x)^{-a-n}\cdot\frac{(-1)^n}{n!}\,\frac{\Gamma(b-a-n)}{\Gamma(c-b-n)}$$

Combined with $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$, this gives the corollary

$\square$

§5. Remarks. - (1) The statement of the corollary above exhibits relation between fundamental solutions of our diff'l eq^n near 0 and ∞.

(2) Heuristically, Γ-weighted inverse Mellin transform

$$\frac{1}{2\pi i} \int_C \varphi(t)\, \Gamma(-t)\, (-x)^t \, dt = \sum_{n=0}^{\infty} \varphi(n) \frac{x^n}{n!}$$

gives a general scheme for writing a formal series as an integral. under certain hypotheses on φ.

Assume φ is hol. on some right half plane
φ doesn't have poles at $\mathbb{Z}_{\geq 0}$
$C: -i\infty \to +i\infty$ s.t. $0, 1, 2, \dots$ is to the right of C and poles of φ are to the left ...

(3) Barnes also obtained the following, via a deformation of contour trick:

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-z)$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; 1+c-a-b; 1-z) \cdot (1-z)^{c-a-b}$$

The proof of the last equation uses Barnes' lemma.

$$\frac{\Gamma(\alpha+\gamma)\,\Gamma(\alpha+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} = \frac{1}{2\pi i}\int_C \Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\gamma-s)\,\Gamma(\delta-s)\,ds$$

§6. Mellin inversion formula.

Let $f : \{\alpha < \mathrm{Re}(z) < \beta\} \to \mathbb{C}$ be a holomorphic function s.t.

$$\int_{-\infty}^{\infty} |f(\sigma+it)|\,dt < \infty$$ and $\lim_{s\to\infty} f(s) = 0$ uniformly for $+\alpha+\delta \leq \mathrm{Re}(s) \leq \beta-\delta$ for any $\delta>0$

Then $g(x) := \dfrac{1}{2\pi i}\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s} f(s)\,ds$ is independent of $\sigma \in (\alpha,\beta)$ and
$(x \in \mathbb{R}_{>0})$

$$f(s) = \int_0^{\infty} g(t)\,t^{s-1}\,dt.$$