Lecture 15

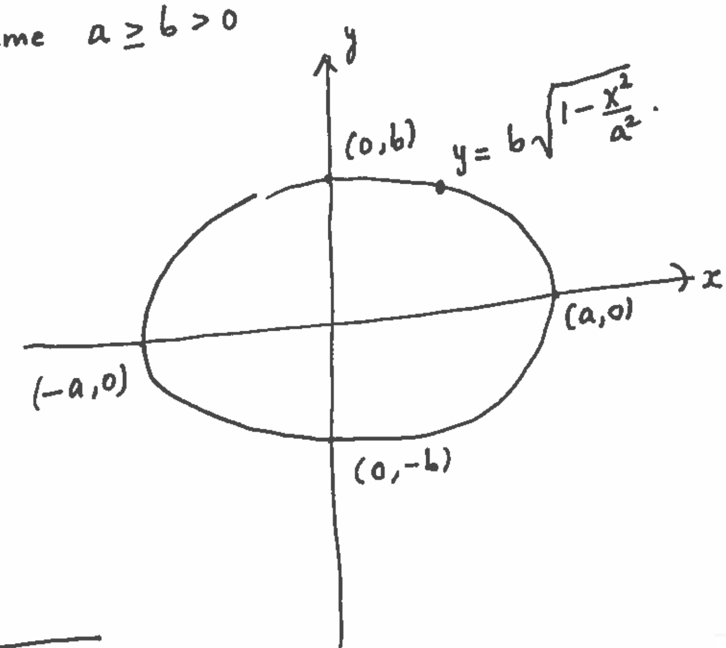
Elliptic integrals.

§1. In general, an elliptic integral is an integral of the form

$$\int_0^x \frac{dt}{\sqrt{R(t)}} \quad , \text{ where degree of } R \text{ is 3 or 4.}$$

Such integrals were first encountered in the problem of finding arc length of an ellipse - hence the name.

Example. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Assume $a \geq b > 0$



Parametrization.

$x = a\cos(\theta)$ $\quad 0 \leq \theta \leq 2\pi$.

$y = b\sin(\theta)$

$\dfrac{dx}{d\theta} = -a\sin(\theta)$ $\quad \dfrac{dy}{d\theta} = b\cos(\theta)$

arc length $\quad L = 4\displaystyle\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

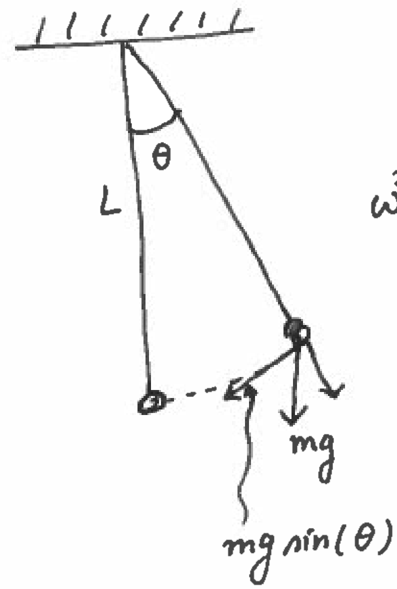
elliptic integral of second kind.

$$= 4a \int_0^{\pi/2} \sqrt{1 - \frac{a^2-b^2}{a^2}\cos^2\theta}\; d\theta = 4a \int_0^1 \frac{1-e^2t^2}{\sqrt{(1-e^2t^2)(1-t^2)}}\, dt$$

say $e^2 = \dfrac{a^2-b^2}{a^2} \in [0,1)$

Another example where elliptic integrals appear is the equation of motion of a pendulum.

$$\theta''(t) = -\omega^2 \sin(\theta(t))$$



$\omega^2 = \frac{g}{L}$

Rewrite as

$$\frac{d}{dt}\left(\frac{\theta'(t)^2}{2} - \omega^2\cos(\theta(t))\right) = 0$$

i.e. $\frac{1}{2}\theta'(t)^2 - \omega^2\cos(\theta(t)) = \underset{\text{constant.}}{C}$

If α is the max. angular displacement, then $\theta'(\alpha) = 0$, so

$C = -\omega^2\cos(\alpha)$.

$$\frac{1}{2}\theta'(t)^2 = \omega^2\cos(\theta(t)) - \omega^2\cos(\alpha).$$

$$\Rightarrow \quad \omega t = \frac{1}{2}\int_0^{\theta(t)} \frac{d\phi}{\sqrt{\sin^2(\frac{\alpha}{2}) - \sin^2(\phi/_2)}}$$

$$= \int_0^{\rho} \frac{dz}{\sqrt{(1-z^2)(1-e^2z^2)}}$$

where $z = \frac{\sin(\phi/2)}{\sin(\alpha/_2)}$

$\rho = \frac{\sin(\theta/2)}{\sin(\alpha/2)}$

$e = \sin\frac{\alpha}{2}$

elliptic integral of first kind.

§2 Gauss' arithmetico-geometric mean.

Let a, b be two positive real numbers. Around 1797, Gauss studied $AGM(a,b)$ defined as follows (this algorithm also appeared in earlier works of Euler and Lagrange).

$$M(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \quad \text{where}$$

$$a_0 = a \qquad a_{n+1} = \frac{a_n + b_n}{2} \qquad \forall\, n \geq 0.$$

$$b_0 = b \qquad b_{n+1} = \sqrt{a_n b_n}$$

<u>Lemma.</u> $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist and are equal. (assume $a \geq b$ for definiteness)

<u>Proof</u>- As $\frac{x+y}{2} \geq \sqrt{xy}$ for $x, y \in \mathbb{R}_{\geq 0}$, $a_n \geq b_n$ $\forall\, n \geq 0$

This implies $a_n \geq \frac{a_n + b_n}{2} = a_{n+1} \geq b_{n+1} = \sqrt{a_n b_n} \geq b_n$.

i.e. $a_0 \geq a_1 \geq a_2 \quad \cdots \quad \geq \quad \cdots\cdots \geq b_1 \geq b_0$

Moreover $a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = \frac{a_n + b_n}{2} - b_n = \frac{a_n - b_n}{2}$

$$\Rightarrow \quad a_n - b_n \leq 2^{-n}(a-b). \qquad \square$$

Theorem (Gauss 1799) : $$\frac{1}{M(a,b)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}$$

Main computationally heavy part of the proof of this theorem consists of checking directly that:

if ϕ and ϕ' are related by
$$\sin\phi = \frac{2a\sin\phi'}{(a+b)+(a-b)\sin^2\phi'},$$
then
$$\frac{d\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}} = \frac{d\phi'}{\sqrt{a_1^2\cos^2\phi'+b_1^2\sin^2\phi'}}$$

This shows $I(a,b) = I(a_1,b_1) = I(a_2,b_2) = \dots = I(\mu,\mu)$

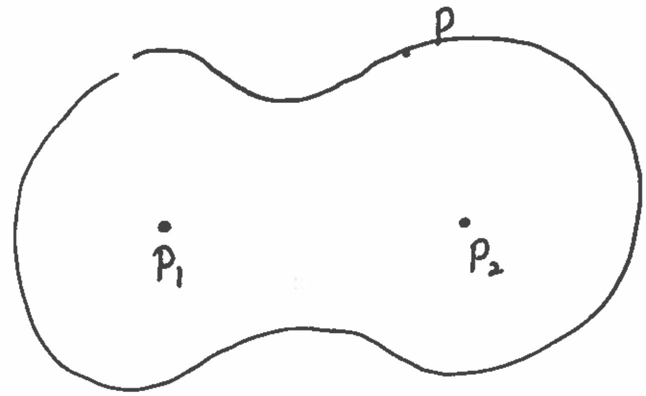
where
$$I(a,b) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}, \quad \mu = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a,b)$$

But $I(\mu,\mu) = \dfrac{\pi}{2\mu}$ and we are done. $\square$

Hint for the computation mentioned above. Setting $b\cdot\tan\phi = t$ gives
$$I(a,b) = \int_0^\infty \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} = \int_0^\infty \frac{dt_1}{\sqrt{(a_1^2+t_1^2)(b_1^2+t_1^2)}}$$
for a suitable change of variables – exercise.

§3. Arc length of Bernoulli's lemniscate.

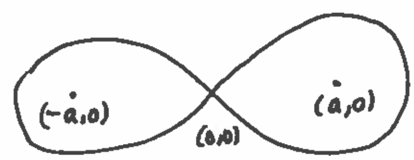
Jacob Bernoulli (1694) discovered the following curve, while solving a mechanical problem of an elastic rod.



general form of "Cassini's Ovals"

$$\{ P : |PP_1| \cdot |PP_2| = \text{constant} \}$$

$P_1 = (-a, 0)$
$P_2 = (a, 0)$
constant $= a^2$



Bernoulli's Lemniscate

$$\left((x-a)^2 + y^2\right)\left((x+a)^2 + y^2\right) = a^4$$

In polar form,

$$r^2 = 2a^2 \cos(2\theta)$$

Take $a = \frac{1}{\sqrt{2}}$ so that we have $r^2 = \cos(2\theta)$.

$$\text{Arc length} \quad L = 4 \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4 \int_0^{\pi/4} \frac{d\theta}{\sqrt{\cos(2\theta)}}$$

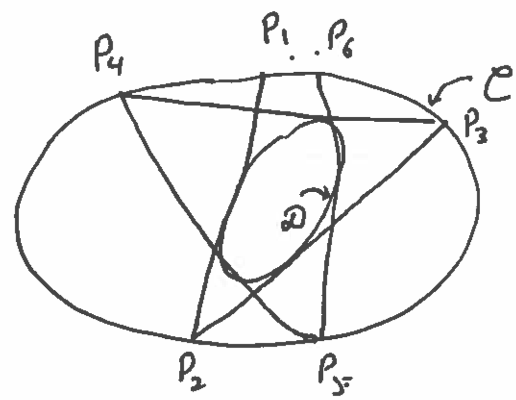
$$= 4 \int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

elliptic integral.

§4. Jacobi's proof of Poncelet's porism for circles.

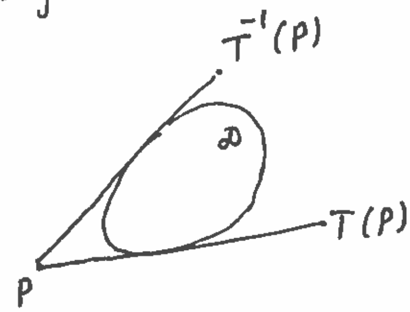
Let $\mathcal{C}$ and $\mathcal{D}$ be two ellipses, $\mathcal{D}$ within $\mathcal{C}$

$P_1, \dots, P_n \in \mathcal{C}$ distinct s.t. $P_i P_{i+1}$ is tangent to $\mathcal{D}$, $P_{n+1} = P_1$, will be called a Poncelet n-gon about $(\mathcal{C}, \mathcal{D})$.



Theorem.- (Poncelet 1822) If a Poncelet n-gon exists about $(\mathcal{C}, \mathcal{D})$, then every point on $\mathcal{C}$ is a vertex of its own Poncelet n-gon.

Written as a map $T: \mathcal{C} \to \mathcal{C}$
$P \mapsto T(P) = P_1$ s.t. $\overrightarrow{PP_1}$ is the right tangent to $\mathcal{D}$



This theorem can be stated as:

if $T^{n+1}(P) = P$ for some $P \in \mathcal{C}$

then $T^{n+1}(P) = P$ for every $P \in \mathcal{C}$.

<u>Remark.-</u> When $\mathcal{C}$ and $\mathcal{D}$ are circles of radii $R > r$ and a = distance between their centers, a result due to Euler and Chapple states that Poncelet triangles about $\mathcal{C}$ and $\mathcal{D}$ exist

$\iff$ $$r = \frac{(R-a)(R+a)}{2R}.$$ This condition is independent of

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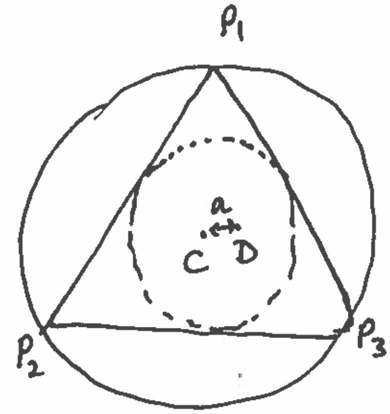
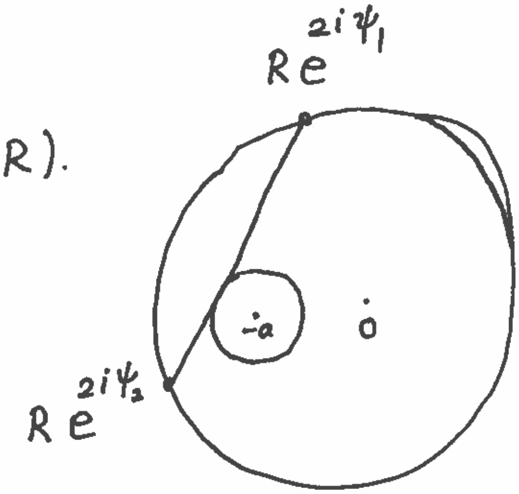
Jean-Victor Poncelet 1788 – 1867

any vertex of the triangle



Jacobi's analytic proof of Poncelet's thm ($\mathcal{C}$ and $\mathcal{D}$ are circles):



Let $\mathcal{C} = C(0;R)$ $\quad (r+a<R)$.

$\mathcal{D} = C(-a;r)$

Let $z_j = Re^{2i\psi_j}$ $(j=1,2)$ be two points on $\mathcal{C}$.

(Jacobi) $z_1 z_2$ is tangent to $\mathcal{D}$ if and only if

$$\boxed{(R+a)\cos\psi_1\cos\psi_2 + (R-a)\sin\psi_1\sin\psi_2 = r}$$

$$\Rightarrow \left(\frac{d\psi_2}{d\psi_1}\right)^2 = \frac{1-k^2\sin^2\psi_2}{1-k^2\sin^2\psi_1} \quad \text{where } k^2 = \frac{4aR}{(R+a)^2-r^2} \in (0,1).$$

i.e. $\dfrac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$ is T-invariant $\quad$ ($T: \mathcal{C} \to \mathcal{C}$ geometrically defined on the previous page)

Jacobi's amplitude function is defined as

$$\mathrm{am} : \mathbb{R} \to \mathbb{R} \qquad \mathrm{am}(0) = 0$$

$$\frac{d}{du}\,\mathrm{am}(u) = \sqrt{1 - k^2 \sin^2(\mathrm{am}(u))}$$

So, if $\psi_1 = \mathrm{am}(u)$, then $\psi_2 = \mathrm{am}(u + c)$ where $\mathrm{am}(c) = \cos^{-1}\left(\frac{r}{R+a}\right)$. Hence $\psi_n = \mathrm{am}(u + nc)$.

This means Poncelet n-gon exists $\iff \psi_n = \psi_1 + \ell\pi$ for some $\ell \in \mathbb{Z}_{\geq 1}$

$\iff \mathrm{am}(u + nc) = \mathrm{am}(u) + \ell\pi. \iff \frac{c}{K} = \frac{2\ell}{n}.$

Lemma. Let $K = \displaystyle\int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$. Then $\mathrm{am}(u + 2K) = \mathrm{am}(u) + \pi$

Proof. - $\mathrm{am}(u) = \varphi$ means $u = \displaystyle\int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$

So $\mathrm{am}(u + 2K) - \mathrm{am}(u)$

$= \varphi_1 - \varphi_2$ where

$$u + 2K = \int_0^{\varphi_1} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \qquad \text{i.e.} \qquad \underbrace{2\int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}}_{} = \int_{\varphi_2}^{\varphi_1} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

$$u = \int_0^{\varphi_2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \qquad = \int_0^{\pi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

gives $\varphi_1 - \varphi_2 = \pi$ $\square$