Lecture 16

Recall - last time we studied various classical problems whose answer is in terms of an elliptic integral $\int \frac{dt}{\sqrt{R(t)}}$.

§1. Angle doubling formula and an "addition theorem"

For the integral $\int_0^x \frac{dt}{\sqrt{1-t^4}}$, the following result was obtained by Fagnano around 1750.

$$2\int_0^u \frac{dt}{\sqrt{1-t^4}} = \int_0^{\frac{2u\sqrt{1-u^4}}{1+u^4}} \frac{dt}{\sqrt{1-t^4}} \quad \text{for } 0<u<\sqrt{\sqrt{2}-1}$$

(for a proof, verify that $f(t) = \dfrac{2t\sqrt{1-t^4}}{1+t^4}$ is monotonically increasing in $[0, \sqrt{\sqrt{2}-1}]$ and $\dfrac{df}{\sqrt{1-f^4}} = \dfrac{dt}{\sqrt{1-t^4}}$.)

Euler's addition theorem.- Let $f(x) = (1-x^2)(1-k^2x^2)$. Then
(1753-1758)

$$\int_0^x \frac{dt}{\sqrt{f(t)}} + \int_0^y \frac{dt}{\sqrt{f(t)}} = \int_0^{A(x,y)} \frac{dt}{\sqrt{f(t)}}, \quad \text{where}$$

$$A(x,y) = \frac{x\sqrt{f(y)} + y\sqrt{f(x)}}{1-k^2x^2y^2}.$$

Sketch of a proof of Euler's theorem.

The level sets of the function $A(x,y) = C$ are defined by solution to

$$\frac{dx}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}} = \frac{dy}{\sqrt{1-y^2}\,\sqrt{1-k^2y^2}} .$$

Replacing this system by

$$\frac{dx}{dt} = \sqrt{(1-x^2)(1-k^2x^2)}$$

$$\frac{dy}{dt} = -\sqrt{(1-y^2)(1-k^2y^2)}$$

, a lengthy, but straightforward calculation gives :

$$\frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = 2k^2xy\,(x^2-y^2) \quad \text{and}$$

$$y^2\left(\frac{dx}{dt}\right)^2 - x^2\left(\frac{dy}{dt}\right)^2 = (y^2-x^2)\,(1-k^2x^2y^2) .$$

This implies

$$\frac{\frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)}{y\frac{dx}{dt} - x\frac{dy}{dt}} = \frac{2k^2xy\left(y\frac{dx}{dt} + x\frac{dy}{dt}\right)}{k^2x^2y^2-1}$$

or

$$y\frac{dx}{dt} - x\frac{dy}{dt} = C\,(k^2x^2y^2-1) \qquad (C \text{ constant})$$

i.e.

$$\frac{x\sqrt{(1-y)(1-k^2y^2)} + y\sqrt{(1-x^2)(1-k^2x^2)}}{1-k^2x^2y^2} = C .$$

$\square$

§2. Periodicity properties of $\int \frac{dt}{\sqrt{1-t^4}}$ were obtained by Gauss in the following manner. Gauss defines "lemniscate sine function" $sl(z)$ by the formula

$$z = \int_0^{sl(z)} \frac{dt}{\sqrt{1-t^4}}$$ and shows that

$$sl(z) = sl(z+\varpi) = sl(z+i\varpi) \quad \text{where} \quad \varpi = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

$$\approx 2.62205\ldots$$

From Gauss' theorem for arithmetic-geometric mean, proved in the previous lecture:

$$M(1,\sqrt{2}) = \frac{\pi}{\varpi}$$

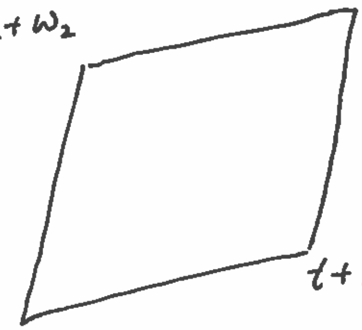
§3. <u>Fundamental theorem of doubly-periodic functions.</u> (see Lecture 37 of part 1)

Let $\varpi_1, \varpi_2 \in \mathbb{C} \setminus \{0\}$ be such that $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$.

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$. A meromorphic function
(period lattice)

$f : \mathbb{C} \dashrightarrow \mathbb{C}$ is called <u>doubly-periodic</u> (with period lattice Λ)

if $f(z+\varpi) = f(z) \ \ \forall\, \omega \in \Lambda$.



(1) If f is holomorphic then f is constant.

(2) Assuming $f \neq$ constant,

f has finitely many zeroes and poles in each parallelogram

$\Pi_t = \{ t + x\omega_1 + y\omega_2 \, ; \, 0 \leq x, y \leq 1 \}$

(assume $t \in \mathbb{C}$ is s.t. f has no zeroes or poles on the boundary of this parallelogram.)

(3) Let $a_1, a_2, \ldots, a_n$ be zeroes of f within Π_t (repeated, according to multiplicity)
$b_1, b_2, \ldots, b_m$ be poles of f within Π_t

Then :

(a) $n = m$

(b) $\displaystyle\sum_{j=1}^{n} \operatorname{Res}_{b_j} f = 0$

(c) $\displaystyle\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j \mod \Lambda.$

(4) Given arbitrary $\{a_j\}$ and $\{b_j\}$ satisfying (a) and (c) of (3) above, there exists a ~~nee~~ doubly periodic $f : \mathbb{C} \dashrightarrow \mathbb{C}$ with zeroes at $\{a_j\}$ and poles at $\{b_j\}$.

§4. Weierstrass' $\wp$-function is a doubly-periodic function with second order pole at each $\omega \in \Lambda$.

Definition : $$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \quad - (*)$$

Theorem.- The series $(*)$ converges uniformly rel. to compact subsets of $\mathbb{C} \setminus \Lambda$. Hence $\wp : \mathbb{C} \dashrightarrow \mathbb{C}$ is a meromorphic function with 2^{nd} order poles at the lattice points Λ.

Proof. - If $K \subset \mathbb{C} \setminus \Lambda$ is a compact set, then we have $\delta > 0$ s.t $|z - \lambda| \geq \delta (>0)$ $\forall z \in K$.

As $$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \lambda^{-2}\left((1 - \lambda^{-1} z)^{-2} - 1 \right)$$
$$= \sum_{n=1}^{\infty} (n+1) z^n \lambda^{-n-2}$$ for $\lambda \in \Lambda$ s.t. $|\lambda| > |z|$ $\forall z \in K$.

~~for each z~~, we can find a constant $C > 0$ (depending only on K)

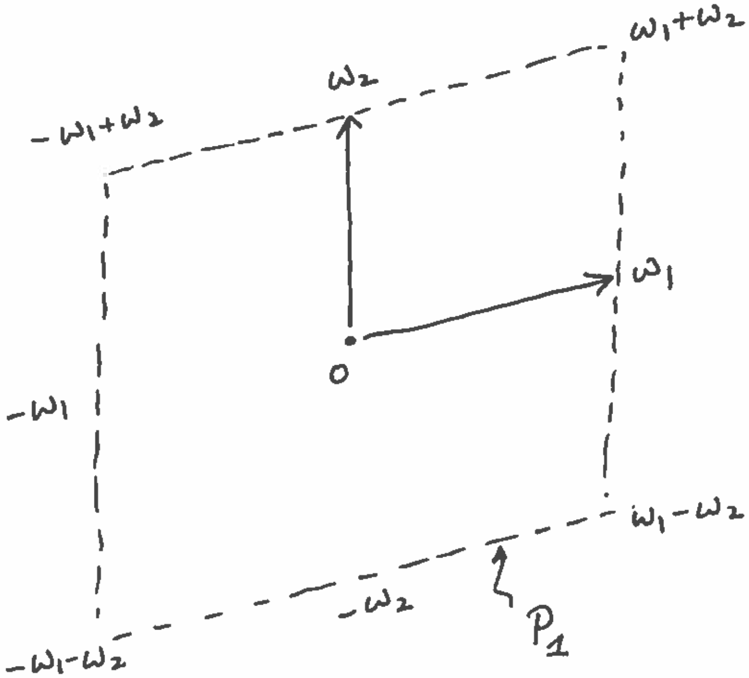
s.t $$\left| \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right| < \frac{C}{|\lambda|^3}$$ $\forall$ $z \in K$ and $\lambda \in \Lambda$ s.t. $|\lambda| > |z'|$ $\forall z' \in K$

The theorem follows from the following general fact:

Lemma. $\displaystyle\sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \frac{1}{\lambda^{\alpha}}$ converges absolutely for $\alpha > 2$.

Let $r =$ smallest distance from 0 to points on the parallelogram drawn here

$R =$ largest ...



$r \le |\lambda| \le R$ for λ on $P_1 \cap \Lambda$.

$|P_1 \cap \Lambda| = 8$

Thus for P_n with vertices $\{\pm n\omega_1 \pm n\omega_2\}$, $|P_n \cap \Lambda| = 8n$ and

$$n r \le |\lambda| \le nR \quad \forall\ \lambda \in |P_n \cap \Lambda|.$$

$$\Rightarrow 8R^{-\alpha}\left(1+\frac{2}{2^{\alpha}}+\dots+\frac{N}{N^{\alpha}}\right) \le \sum_{\substack{\lambda\in\Lambda\cap P_n \\ n=1,2,\dots,N}} \frac{1}{|\lambda|^{\alpha}} \le 8\cdot r^{-\alpha}\cdot\left(1+\frac{2}{2^{\alpha}}+\frac{3}{3^{\alpha}}+\dots+\frac{N}{N^{\alpha}}\right)$$

$$\le \frac{8}{r^{\alpha}}\left(\sum_{k=1}^{\infty}\frac{1}{k^{\alpha-1}}\right)$$

converges for $\alpha > 2$.

$\square$

§5. Properties of $\wp(z)$.

(1) $$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}$$ is an odd doubly periodic function of period lattice λ. It has ord. 3 pole at points of Λ and no other poles.

(2) $\wp(z)$ is an even function. $\wp(z+\omega_1) = \wp(z) = \wp(z+\omega_2)$

(Proof.– $\wp(z+\omega_1)$ and $\wp(z)$ have the same derivative

So $\wp(z+\omega_1) - \wp(z) = A$ some constant.

Set $z = -\frac{\omega_1}{2}$, use the fact that $\wp$ is even, to conclude $A = 0$. $\square$)

(3) $\wp'(z)^2 = 4\,\wp(z)^3 - g_2\,\wp(z) - g_3$ where

$$g_2 = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-4} \qquad g_3 = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-6}$$

Proof. $$\wp(z) - z^{-2} = \sum_{\lambda \in \Lambda^\circ} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \qquad (\Lambda^\circ = \Lambda \setminus \{0\})$$

$\cancel{\sum_{\lambda \in \Lambda^\circ}}$ is an even function holomorphic near 0, so its Taylor series expansion

$$= c_2 z^2 + c_4 z^4 + \cdots$$

where $$2c_2 = \frac{d^2}{dz^2}\left(\wp^{\#}(z) - z^{-2}\right)\Big|_{z=0}$$

$$= \sum_{\lambda\in\Lambda^0} \frac{6}{(z-\lambda)^4}\Big|_{\lambda=0} = 6\sum_{\lambda\in\Lambda^0} \lambda^{-4}$$

So, $c_2 = g_2/20$. Similarly $c_4 = \frac{g_3}{28}$.

Hence we get $$\wp(z) = z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + \cdots$$

$$\wp'(z) = -2z^{-3} + \frac{1}{10} g_2 z + \frac{1}{7} g_3 z^3 + \cdots$$

$$\wp(z)^3 = z^{-6} + \frac{3}{20} g_2 z^{-2} + \frac{3}{28} g_3 + \cdots$$

$$\wp'(z)^2 = 4z^{-6} - \frac{2}{5} g_2 z^{-2} - \frac{4}{7} g_3 + \cdots$$

$$\Rightarrow \quad \wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3 = O(z) \quad (\text{i.e. } = 0 \text{ at } z=0)$$

and is doubly-periodic hence a constant $= 0$. $\square$