Lecture 18

§1. Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e. rank 2 subgroup. Recall, last time, we encountered a few numerical invariants of Λ

$$g_2 = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-4} \qquad g_3 = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-6}$$

$$\Delta = g_2^3 - 27 g_3^2 \qquad J = \frac{g_2^3}{\Delta}$$

<u>Note</u> if we scale Λ, these invariants change by

$$g_2(\alpha\Lambda) = \alpha^{-4} g_2(\Lambda) \qquad g_3(\alpha\Lambda) = \alpha^{-6} g_3(\Lambda)$$

$$\Delta(\alpha\Lambda) = \alpha^{-12} \Delta(\Lambda) \quad \text{and} \quad J(\alpha\Lambda) = J(\Lambda).$$

Thus if ω_1, ω_2 are generators of Λ such that $\operatorname{Im}(\omega_2/\omega_1) > 0$ ($\tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}$ upper half plane)

then, for instance

$$g_2(\omega_1, \omega_2) = \omega_1^{-4} g_2(1, \tau)$$

and so on. Note $J(\omega_1, \omega_2) = J(1, \tau)$ usually denoted by $j(\tau)$.

§2. Lemma. - Let $\{\omega_1, \omega_2\}$ and $\{\omega_2' = a\omega_2 + b\omega_1, \omega_1' = c\omega_2 + d\omega_1\}$ $(a, b, c, d \in \mathbb{Z})$

be two sets of generators of Λ s.t.

$$\tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad \tau' = \frac{\omega_2'}{\omega_1'} \quad \text{are in } \mathbb{H}$$

Then $ad - bc = 1$. i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

Proof.- If $\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$ gives another set of generators of Λ, then $ad - bc = \pm 1$. Now,

$$\tau' = \frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1} = \frac{a\tau + b}{c\tau + d}$$

$$\Rightarrow \quad \mathrm{Im}(\tau') = \frac{ad - bc}{|c\tau + d|^2} \mathrm{Im}(\tau). \quad \text{So, } ad - bc = 1.$$

□

Cor. $j\left(\frac{a\tau + b}{cz + d}\right) = j(\tau) \quad \forall \; a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$.

§3. Möbius transformations - recap.

$A \in GL_2(\mathbb{C}) \rightsquigarrow f(z) = \frac{az + b}{cz + d}$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

denoted by M_A

(a) $M_{\lambda A} = M_A$, $M_{Id} = Id \; (z \mapsto z)$, $M_{AB} = M_A \circ M_B$.

(b) $M_A'(z) = \frac{\det(A)}{(cz + d)^2}$. So M_A preserves angles (except for the pole $-d/c$)

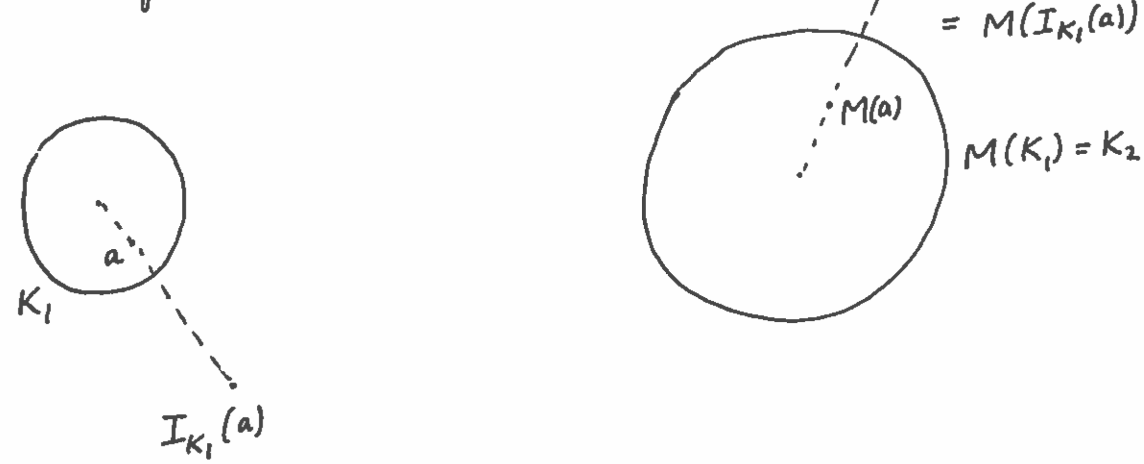
(c) Each M_A $(A \neq \lambda \cdot Id)$ has 1 or 2 fixed points.

(d) M_A maps circles (lines considered as circles through ∞ on $\mathbb{P}^1$) to circles.

(e) $\forall\ \alpha, \beta, \gamma \in \mathbb{P}^1$ distinct, $\exists!$ Möbius transformation

$$M : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \quad \text{s.t.} \quad \begin{aligned} M(\alpha) &= 0 \\ M(\beta) &= 1 \\ M(\gamma) &= \infty . \end{aligned}$$

(f) Möbius transformations commute with inversion w.r.t. circles



§4. Upper half plane and $SL_2(\mathbb{Z})$ - action.

$\mathbb{H} = \{ z : Im(z) > 0 \}$

$SL_2(\mathbb{Z}) \circlearrowright \mathbb{H}$ by $\tau \mapsto \dfrac{a\tau + b}{c\tau + d}$. Note $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ act the same way.

Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : \tau \mapsto \dfrac{-1}{\tau}$

$T = \begin{bmatrix} \phi & 1 \\ 0 & 1 \end{bmatrix} : \tau \mapsto \tau + 1.$

Theorem. – (1) $SL_2(\mathbb{Z})$ is generated by S and T.

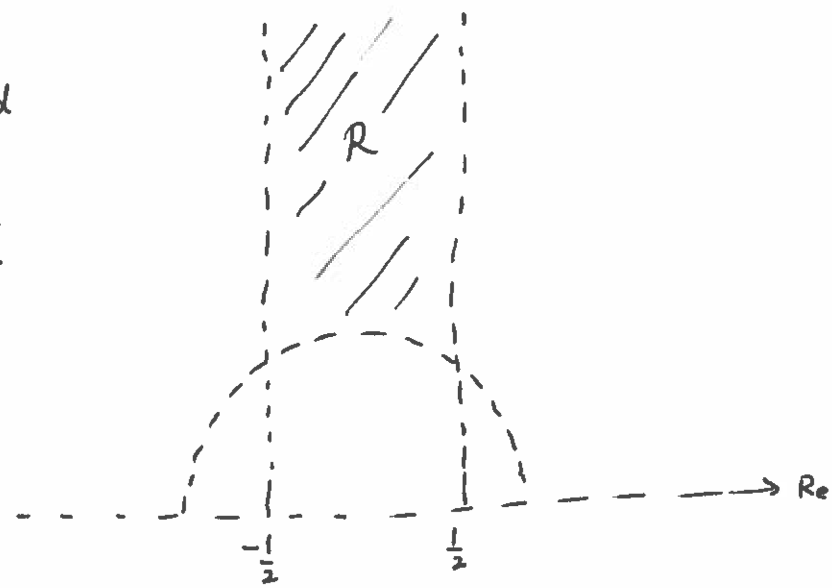
(NOT freely, there are relations)

(2) Let $R \subset \mathbb{H}$ consist of τ s.t. $-\frac{1}{2} < Re(\tau) < \frac{1}{2}$ and $|\tau| > 1$.

Then $M_A(\tau) = \tau'$

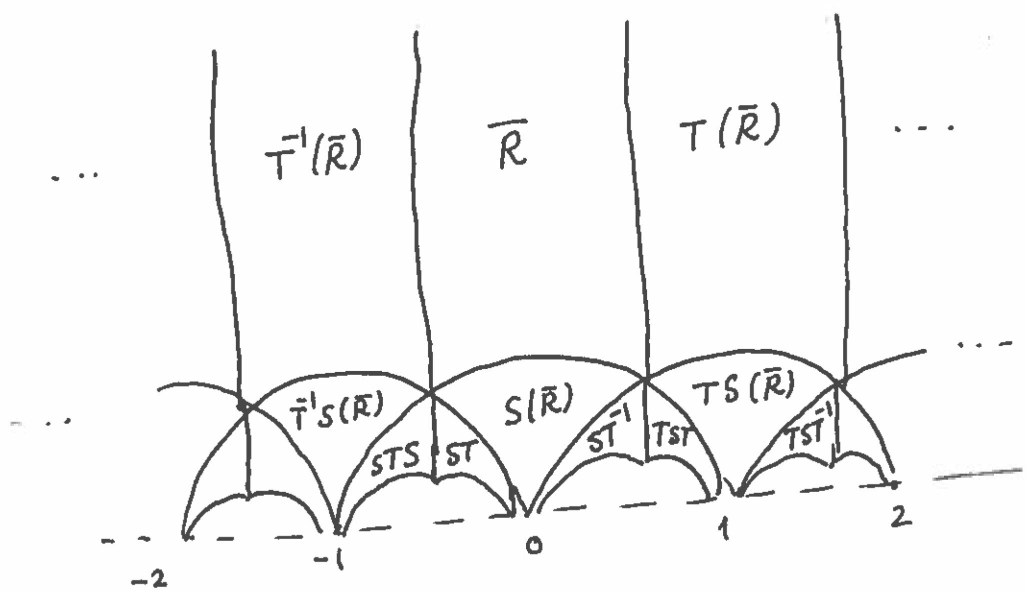
for some $A \in SL_2(\mathbb{Z})$ and $\tau, \tau' \in R$

$\Rightarrow A = \pm Id$ and $\tau = \tau'$.



(3) $$\mathbb{H} = \bigcup_{A \in SL_2(\mathbb{Z})/_{\pm 1}} M_A(\bar{R})$$

(the union is disjoint – except for the boundary)



Proof of (1). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

If $c = 0$, then $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = T^b$.

(up to -1)

If $c = 1$, then $A = \begin{bmatrix} a & ad-1 \\ 1 & d \end{bmatrix} = T^a S T^d$.

Now we argue that using S and T we can make c smaller

Assume $c \geq 2$. $ad - bc = 1 \Rightarrow \gcd(c,d) = 1$ i.e.,

$d = c \cdot q + r$, $0 < r < c$.

$$A \cdot T^{-q} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -q \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -aq+b \\ c & r \end{bmatrix}$$

$$A T^{-q} S = \begin{bmatrix} -aq+b & -a \\ r & -c \end{bmatrix}$$

as claimed, (2,1) entry $r \in \{1, 2, \ldots, c-1\}$.

$\square$

Proof of (2). Note that if $\tau' = \dfrac{a\tau + b}{c\tau + d}$, then

$$\operatorname{Im}(\tau') = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}. \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Now, if $\tau \in R$ and $c \neq 0$, then $|c\tau + d|^2 > 1$

$$\left(|c\tau + d|^2 = c^2|\tau|^2 + d^2 + cd(\tau + \bar{\tau}) \underset{\substack{\uparrow \\ |\tau| > 1 \\ |\tau + \tau'| < \frac{1}{2}}}{>} c^2 + d^2 - |cd| \right)$$

So, if $\tau, \tau' \in R$, then $\quad Im(\tau') < Im(\tau)$ and $Im(\tau) < Im(\tau')$.

(and $\tau' = \dfrac{a\tau+b}{c\tau+d}$, $\tau = \dfrac{d\tau'-b}{-c\tau'+a}$)

with $c \neq 0$

This means $c = 0$ and $A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ $n \in \mathbb{Z}$

$\Rightarrow \quad \tau' = \tau + n$. But $-\frac{1}{2} < Re(\tau), Re(\tau') < \frac{1}{2} \Rightarrow n = 0$

and hence $\tau = \tau'$.

Proof of (3). From (2), it is clear that $R \cap M_A(R) = \phi$.
$(\forall A \in SL_2(\mathbb{Z}),\ A \neq \pm Id)$

It remains to see that every $\tau \in \mathbb{H}$ can be brought to $\overline{R}$ by some element of $SL_2(\mathbb{Z})$.
The proof is given in the next section, and is based on a geometric description of $\mathbb{Z}$- basis of a lattice.

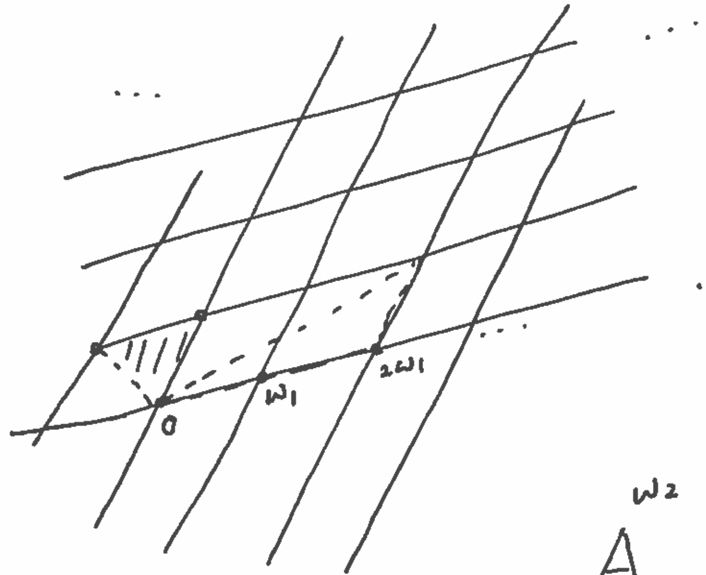
§5. Bases of lattice $\Lambda \subset \mathbb{C}$.

Lemma. $\{\omega_1, \omega_2\} \subset \Lambda$ is a $\mathbb{Z}$- basis of Λ
$\Leftrightarrow$ The triangle with vertices $0, \omega_1, \omega_2$ contains no points of Λ (except for $0, \omega_1$ & ω_2).

Proof. - If $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ then the parallelogram with vertices $0, \omega_1, \omega_2, \omega_1+\omega_2$ is given by

$$\{t_1\omega_1 + t_2\omega_2 : 0 \le t_1, t_2 \le 1\}$$

which intersects Λ only at $0, \omega_1, \omega_2$ and $\omega_1+\omega_2$.



Conversely, assume $T \cap \Lambda = \{0, \omega_1, \omega_2\}$ $\quad T =$

Let $\omega \in \Lambda$. Then $\omega \equiv r_1\omega_1 + r_2\omega_2 \mod \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \Lambda$.

$$0 \le r_1, r_2 < 1$$

If r_1 or r_2 is non-zero we have an element of Λ in the parallelogram $0, \omega_1, \omega_2, \omega_1+\omega_2$, hence one in the triangle T.

So, $r_1 = r_2 = 0$ and $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \Lambda$. $\square$

Returning to the proof of Thm §4, (3), let $\tau \in \mathbb{H}$ and let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Let $\tau_1 \in \Lambda$ be such that $\tau_1 \in \mathbb{H}$ and $|\tau_1|$ is smallest among $\{|w| : w \in \Lambda \setminus \mathbb{Z}\}$

Then $|\tau_1 \pm 1| \ge |\tau_1|$

Proof of Thm §4 (3). We have to show that $\forall\, \tau \in \mathbb{H}$,

$\exists\, A \in SL_2(\mathbb{Z})$ s.t. $M_A(\tau) = \tau' \in R$ i.e., $|\tau'| \geq 1$

and $|Re(\tau')| \leq \frac{1}{2}$.

Let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Arrange elements of Λ according to increasing modulus and then argument

$$\Lambda = \{0, w_1, w_2, \dots\} \quad \text{s.t.} \quad |w_1| \leq |w_2| \leq \cdots$$

and if $|w_j| = |w_{j+1}|$ then

$\arg(w_j) < \arg(w_{j+1})$

$(0 \leq \arg < 2\pi)$.

Take $\omega_1 = w_1$

and ω_2 = first element of $\Lambda \setminus \mathbb{Z}\omega_1$.

Then, by lemma above, $\{\omega_1, \omega_2\}$ give a basis of Λ

Moreover $|\omega_2| \geq |\omega_1|$ $\Rightarrow$ $|\tau'| \geq 1$ $\left(\tau' = \frac{\omega_2}{\omega_1}\right)$

and $|\omega_2 \pm \omega_1| \geq |\omega_2|$ $\quad$ $|\tau' \pm 1| \geq |\tau'|$

$\omega_2 = a\tau + b$

$\omega_1 = c\tau + d$ $\Rightarrow$ $\tau' = M_A(\tau)$ and $|x \pm 1| \geq |x|$ is same as $|Re(x)| \leq \frac{1}{2}$.

$\square$