Lecture 19

Recall - We introduced $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{matrix} \right\}$

action on the upper half-plane $\mathbb{H} = \{ z \mid Im(z) > 0 \}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau+b}{c\tau+d} = \tau' \quad \left(Im(\tau') = \frac{Im(\tau)}{|c\tau+d|^2} \right).$$

Fundamental region for $SL_2(\mathbb{Z}) \circlearrowright \mathbb{H}$:

$$R = \left\{ \tau \in \mathbb{H} \mid -\tfrac{1}{2} < Re(\tau) < \tfrac{1}{2} \text{ and } |\tau| > 1 \right\}$$

§1.

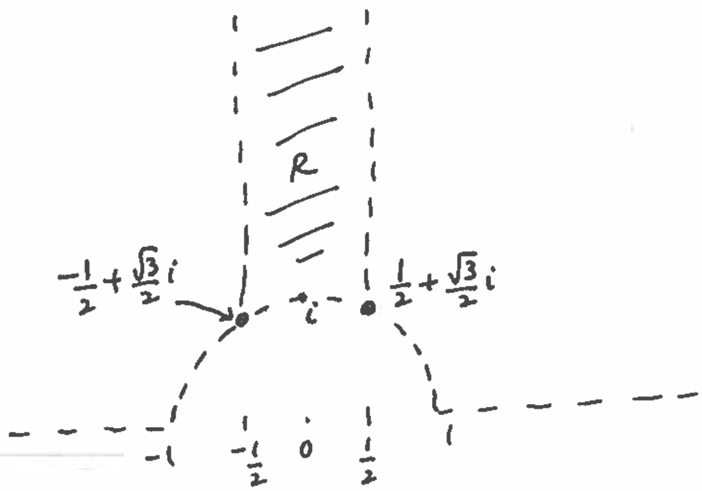
For a lattice $\Lambda \subset \mathbb{C}$,

let $\{\omega_1, \omega_2\}$ and $\{\omega_1', \omega_2'\}$

be two sets of generators;

ordered s.t.

$Im(\omega_2/\omega_1),\ Im(\omega_2'/\omega_1') > 0$



Then $\tau' = \dfrac{\omega_2'}{\omega_1'} = \dfrac{a\tau+b}{c\tau+d}$ $(\tau = \omega_2/\omega_1)$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-4} \qquad g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-6}.$$

or $g_2(\omega_1, \omega_2)$ $\qquad g_3(\omega_1, \omega_2)$

$$\Delta = g_2^3 - 27 g_3^2 \quad \text{and} \quad J = \frac{g_2^3}{\Delta} .$$

We will write $g_2(\tau)$, $g_3(\tau)$ etc. for $g_2(1,\tau)$, $g_3(1,\tau)$ etc.

Note: if $G_{2\ell}(\omega_1, \omega_2) = \sum\limits_{\substack{(m,n) \neq (0,0) \\ m,n \in \mathbb{Z}}} \frac{1}{(m\omega_1 + n\omega_2)^{2\ell}}$, then

$$G_{2\ell}(\omega_1, \omega_2) = \omega_1^{-2\ell} G_{2\ell}(1, \tau) . \text{ So, if we set}$$

$G_{2\ell}(\tau) = G_{2\ell}(1,\tau)$, then

$$\boxed{G_{2\ell}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2\ell} G_{2\ell}(\tau)}$$

(Proof - $\{\omega_1, \omega_2\} \xrightarrow[\begin{pmatrix} a & b \\ c & d \end{pmatrix}]{} \{\omega_1', \omega_2'\}$ $\quad \begin{aligned} \omega_2' &= a\omega_2 + b\omega_1 \\ \omega_1' &= c\,\omega_2 + d\omega_1 \end{aligned}$

generate the same lattice, hence $G_{2\ell}(\omega_1, \omega_2) = G_{2\ell}(\omega_1', \omega_2')$

$$\Rightarrow \quad G_{2\ell}(\tau') = \frac{(\omega_1')^{2\ell}}{\omega_1^{2\ell}} G_{2\ell}(\tau)$$

$$= (c\tau + d)^{2\ell} G_{2\ell}(\tau) . \qquad \square \;)$$

Note: $\quad j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau) \quad \forall\, \tau \in \mathbb{H}, \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

§2. Definition. - A <u>modular function</u> is a meromorphic

$f : \mathbb{H} \dashrightarrow \mathbb{C}$ s.t.

(1) $$f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \quad \forall\, \tau \in \mathbb{H}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

(2) f has at most a pole at $i\infty$.

<u>Meaning of item (2).</u> Take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to get, from (1),

$$f(\tau+1) = f(\tau).$$

Hence f is a function of $e^{2\pi i \tau} = x$; $\quad 0 < |x| < 1$.

The point $x = 0$ corresponds to the limit $Im(\tau) \to \infty$.

Condition (2) requires Laurent series expansion of $f(x)$ around $x = 0$ to have polynomial singular part:

$$f(x) = \sum_{n=-M}^{\infty} c_n x^n .$$

We will call the expansion $f(\tau) = \sum_{n=-M}^{\infty} c_n e^{2\pi i n \tau}$, Fourier expansion of modular function f, near $i\infty$.

§3. <u>Fourier expansion of g_2, g_3, Δ and j</u>

g_2, g_3 and Δ are not modular functions, as defined above, but have weight 2, 3 and 6 respectively - i.e.,

$$g_2(\tau') = (c\tau+d)^4 g_2(\tau) \qquad g_3(\tau') = (c\tau+d)^6 g_3(\tau)$$

and $$\Delta(\tau') = (c\tau+d)^{12}\Delta(\tau) \qquad \left(\tau' = \frac{a\tau+b}{c\tau+d}\right).$$

So, (taking $a=b=1$, $c=0$, $d=1$) these functions of $\tau \in \mathbb{H}$ are still 1-periodic, and hence admit Fourier expansion $\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n\tau}$.

Theorem.-
$$g_2(\tau) = \frac{4\pi^4}{3}\left(1 + 240\sum_{k=1}^{\infty}\sigma_3(k)\, e^{2\pi i k\tau}\right)$$

$$g_3(\tau) = \frac{8\pi^6}{27}\left(1 - 504\sum_{k=1}^{\infty}\sigma_5(k)\, e^{2\pi i k\tau}\right)$$

where $\sigma_a(k) = \sum_{d|k} d^a$.

Remark.- More generally for Eisenstein series $G_{2\ell}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+n\tau)^{-2\ell}$

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{(-1)^k 2\cdot(2\pi)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\, e^{2\pi i n\tau}$$

Proof.- We first show, using partial fraction expansion of

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\left(\frac{1}{x+m} - \frac{1}{m}\right)$$

$$\frac{1}{\tau} + \sum_{m \in \mathbb{Z}_{\neq 0}} \left(\frac{1}{\tau+m} - \frac{1}{m} \right) = -\pi i \left(1 + 2 \sum_{r=1}^{\infty} e^{2\pi i r \tau} \right) - (*)$$

that, for each $n > 0$, we have:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^4} = \frac{8\pi^4}{3} \sum_{r=1}^{\infty} r^3 e^{2\pi i r n \tau} \quad - (1)$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^6} = -\frac{8\pi^6}{15} \sum_{r=1}^{\infty} r^5 e^{2\pi i r n \tau} \quad - (2)$$

(Proof: differentiate (*) repeatedly to get

$$-\sum_{m=-\infty}^{\infty} \frac{1}{(\tau+m)^2} = -(2\pi i)^2 \sum_{r=1}^{\infty} r e^{2\pi i r \tau}$$

$$-3! \sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^4} = -(2\pi i)^4 \sum_{r=1}^{\infty} r^3 e^{2\pi i r \tau}$$

$$-5! \sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^6} = -(2\pi i)^6 \sum_{r=1}^{\infty} r^5 e^{2\pi i r \tau}. \;)$$

Now,

$$g_2(\tau) = 60 \left(\sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m^4} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \left(\frac{1}{(m+n\tau)^4} + \frac{1}{(m-n\tau)^4} \right) \right)$$

$$= 60 \left(2\zeta(4) + 2 \sum_{n=1}^{\infty} \left(\frac{8\pi^4}{3} \sum_{r=1}^{\infty} r^3 e^{2\pi i r n \tau} \right) \right)$$

$$= 120 \cdot \frac{\pi^4}{90} + 320\pi^4 \sum_{N=1}^{\infty} e^{2\pi i N \tau} \left(\sum_{r \mid N} r^3 \right) \qquad \square$$

§4. Corollary.

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \underbrace{\tau_R(n)}_{\text{Ramanujan's } \tau\text{-function}} e^{2\pi i n \tau}$$

where $\tau_R(n)$ are integers.

Proof.- Let $q = e^{2\pi i \tau}$, $A = \sum_{n=1}^{\infty} \sigma_3(n) x^n$, $B = \sum_{n=1}^{\infty} \sigma_5(n) x^n$. Then,

$$\Delta(\tau) = g_2(\tau)^3 - 27 g_3(\tau)^2 = \frac{64\pi^{12}}{27} \left\{ (1+240A)^3 - (1-504B)^2 \right\}$$

$$= \frac{64\pi^{12}}{27} \left(12^2 (5A+7B) + 12^3 (100A^2 - 147B^2 + 8000A^3) \right)$$

Note : $5d^3 + 7d^5 \equiv 0 \pmod{12}$ $\forall d \in \mathbb{Z}_{\geq 1}$,

So $5A + 7B \equiv 0 \mod 12$

$$\Rightarrow \Delta(\tau) = \frac{64\, \pi^{12}\, 12^3}{27} (\text{ series with integer coefficients}).$$

$$= (2\pi)^{12} \left(\sum_{n=1}^{\infty} \tau_R(n)\, e^{2\pi i n \tau} \right). \qquad \square$$

Ex. $\tau_R(1) = 1$, $\tau_R(2) = -24$.

§5. <u>Cor.</u> - $$12^3 j(\tau) = e^{-2\pi i \tau} + 744 + \sum_{n=1}^{\infty} c(n)\, e^{2\pi i n \tau}$$

where $c(n) \in \mathbb{Z} \quad \forall\, n \geq 1.$

Proof.- $$g_2(\tau)^3 = \frac{64}{27}\pi^{12}\left(1 + 240\,x + I\right)^3 \qquad x = e^{2\pi i \tau}$$

$$= \frac{64}{27}\pi^{12}\left(1 + 720\,x + I_1\right)$$

and $$\Delta(\tau) = \frac{64}{27}\pi^{12}\left(12^3 x\,\left(1 - 24x + J\right)\right)$$

(Here, I, I_1 and J are series with integer coefficients.)

Hence, $$j(\tau) = \frac{1 + 720\,x + I_1}{12^3 x\,(1 - 24x + J)}$$

$$= \frac{1}{12^3 x}\left(1 + 720\,x + I_1\right)\left(1 + 24x + J'\right)$$

$\Rightarrow$ $j(\tau)\cdot 12^3 = \dfrac{1}{x} + 744 +$ series with integer coefficients. $\square$