Lecture 20

Recall: $\mathbb{H}$ = upper half plane $= \{z \mid Im(z) > 0\}$

$SL_2(\mathbb{Z}) \circlearrowright \mathbb{H}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \dfrac{a\tau + b}{c\tau + d}$.

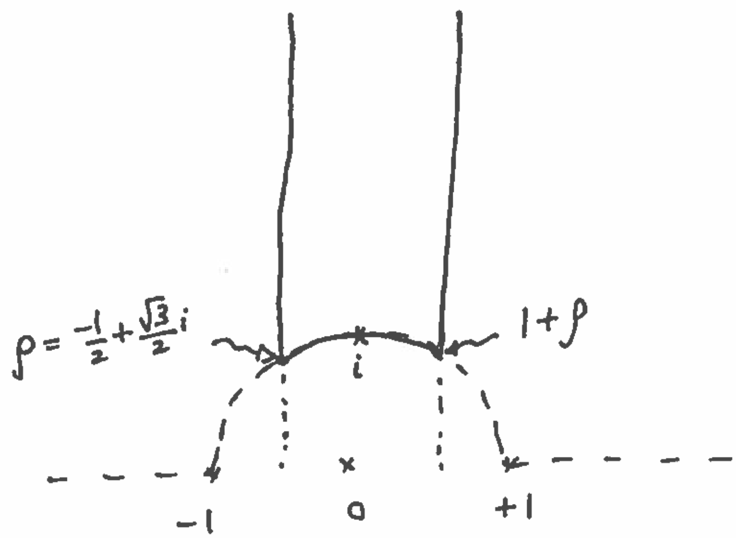
$R \subset \mathbb{H}$ is given by $R = \left\{ \tau \in \mathbb{H} \;\middle|\; \begin{matrix} -\frac{1}{2} < Re(\tau) < \frac{1}{2} \\ |\tau| > 1 \end{matrix} \right\}$

<u>Ex.</u>

Stabilizer of $\rho = e^{\frac{2\pi i}{3}}$ consists of

Id, $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

(up to ± 1)

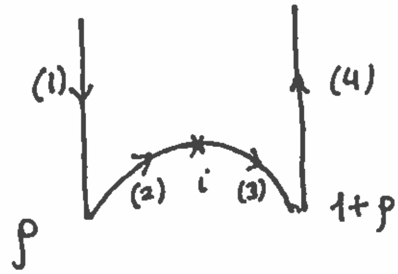


Stabilizer of i consists of Id, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (up to ± 1).

§1. <u>Conventions for counting zeroes and poles within $\overline{R}$.</u>

Assume $f : \mathbb{H} \dashrightarrow \mathbb{C}$ is a meromorphic function which is invariant under $SL_2(\mathbb{Z})$-action and has at most a pole at $i\infty$. [We will say, in this case, that f is a <u>modular function</u>.]

If f has a zero or pole on the edge (1) or (2), then it also has a zero/pole at the equivalent point of (4) or (3).



Boundary ∂R

We only consider zeroes/poles on (1) and (2) as "belonging to $\overline{R}$".

The order of zero/pole at ρ is to be divided by $3 = |\text{Stabilizer of } \rho|$

The order of zero/pole at i is to be divided by $2 = |\text{Stabilizer of } i|$.

Behaviour at $i\infty(\tau) =$ Behaviour at 0 $(= e^{2\pi i\tau})$.

§2. <u>Theorem</u>. – Assume f is a modular function. Then, assuming f is not identically zero, the number of zeroes of f within $\overline{R}$ = number of poles within $\overline{R}$.

§3. An <u>entire modular form of weight k</u> is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, analytic at $i\infty$, such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$
$$\tau \in \mathbb{H}.$$

<u>Theorem.</u> – Assume f is an entire modular form of weight k, not identically zero. Then

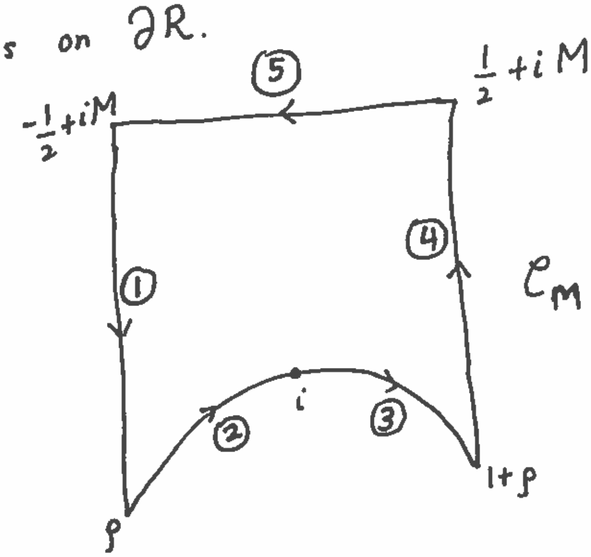
$$\boxed{k = 12N + 6N(i) + 4N(\rho) + 12N(i\infty)}$$

Here, $N =$ number of zeroes of f in $\overline{R}$ (omitting the vertices)

$N(p) =$ order of vanishing of f at p

§4. Proof of Thm §2. –

<u>Case 1.</u> – f has no zeroes or poles on ∂R.



Then

$$Z - P = \frac{1}{2\pi i}\int_{\mathcal{C}_M} \frac{f'(z)}{f(z)}\,dz \quad (M \to \infty)$$

(here $Z =$ number of zeroes of f in R
$P =$ number of poles of f in R).

Note: as $i\infty$ is assumed to be an isolated singularity, and $f \not\equiv 0$, $\exists\ M > 0$ s.t. zeroes / poles of f in R lie within the contour $\mathcal{C}_M$.

As $f(z+1) = f(z)$ and $f\left(\frac{-1}{z}\right) = f(z)$, integrals over

1 and 4 ; respectively 2 and 3 cancel each other, and

we are left with

$$Z - P = \lim_{M\to\infty} \frac{-1}{2\pi i} \int_{-\frac{1}{2}+Mi}^{\frac{1}{2}+Mi} \frac{f'(z)}{f(z)}\, dz$$

Set $w = e^{2\pi i z}$, so that $z = t + Mi$ $\left(-\frac{1}{2} \le t \le \frac{1}{2}\right)$ gets mapped

to $w = e^{-2\pi M} \cdot e^{2\pi i t}$ (circle of radius $e^{-2\pi M}$ centered at 0).

Moreover, if $f(z) = \sum_{n=-\ell}^{\infty} a_n e^{2\pi i n z}$ $(\ell \in \mathbb{Z})$, and $a_{-\ell} \neq 0$

we write $F(w) = \sum_{n=-\ell}^{\infty} a_n w^n$, then $\frac{f'(z)}{f(z)}\, dz = \frac{F'(w)}{F(w)}\, dw$

So, $Z - P = \dfrac{-1}{2\pi i} \displaystyle\oint_{0} \frac{F'(w)}{F(w)}\, dw = \ell$.

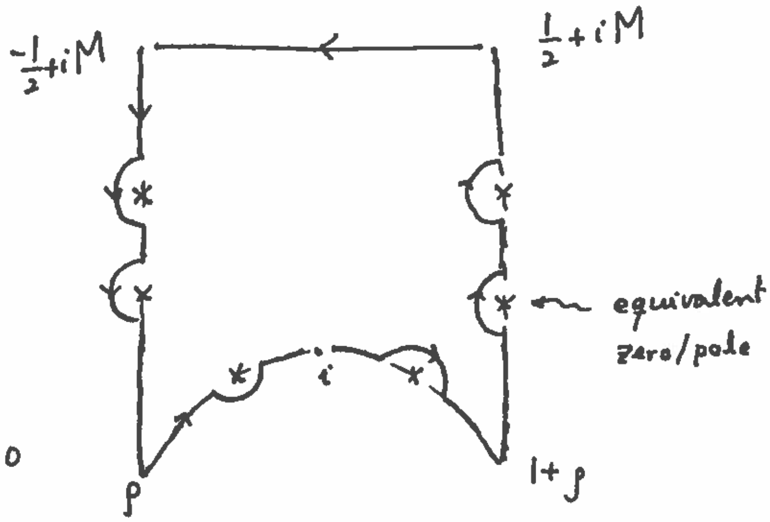
If $\ell > 0$, then we have a pole at $i\infty$ of order ℓ, so $Z = P + \ell$ as claimed in Thm §2.

If $\ell < 0$, then we have a zero at $i\infty$ of order $-\ell$, so, $Z - \ell = P$ as claimed

This finishes the proof of the theorem, in the case when f has no zeroes / poles on ∂R.

<u>Case 2.</u> – Next we consider the situation where f has zeroes or poles on ∂R – but not at the vertices ρ, i, $1+\rho$.

The same argument as in case 1 applies to indented contour $\mathcal{C}_M$ and the result follows.



It remains to compute $\int \frac{f'(x)}{f(x)}\,dx$ when f has zeroes/poles at ρ, i.

Assume $f(z) = (z-\rho)^n\, g(z)$; $g(\rho) \neq 0$, $n \in \mathbb{Z}$.

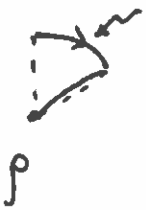
<u>Calculation near ρ</u> :

$$\frac{1}{2\pi i}\int_{\mathcal{A}^{(r)}} \frac{f'(z)}{f(z)}\,dz$$

$$= \frac{1}{2\pi i}\int_{\mathcal{A}^{(r)}} \left(\frac{n}{z-\rho} + \frac{g'(z)}{g(z)}\right) dz$$

$$= \frac{1}{2\pi i}\int_{\mathcal{A}^{(r)}} \frac{n}{z-\rho}\,dz + \frac{1}{2\pi i}\int_{\mathcal{A}^{(r)}} \frac{g'(z)}{g(z)}\,dz$$

$\mathcal{A}^{(r)} =$ arc $r \cdot e^{i\theta} + \rho$



θ decreases from $\frac{\pi}{2}$ to $\alpha(r)$

$\lim_{r \to 0^+} \alpha(r) =$ angle between y-axis and the tangent to $C(0;1)$ at ρ

$= \frac{\pi}{6}$.

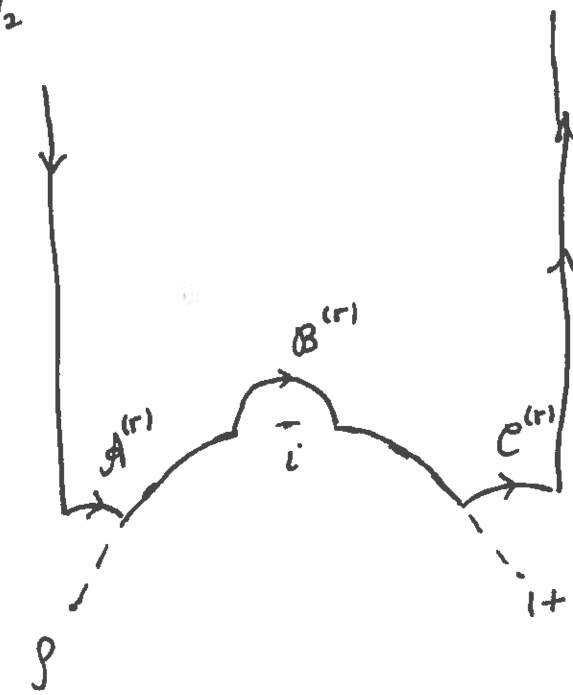
As $r \to 0^+$, $\left| \frac{g'(z)}{g(z)} \right|$ remains bounded $(g(\rho) \neq 0)$

So $$\left| \int_{A^{(r)}} \frac{g'(z)}{g(z)}\, dz \right| \leq \text{length}(A^{(r)}) \cdot M \to 0 \text{ as } r \to 0.$$

some constant

$$\Rightarrow \lim_{r \to 0^+} \frac{1}{2\pi i} \int_{A^{(r)}} \frac{f'(z)}{f(z)}\, dz = \lim_{r \to 0^+} \frac{n}{2\pi i} \int_{A^{(r)}} \frac{dz}{z-\rho}$$

$$\underset{(z=\rho+re^{i\theta})}{=} \lim_{r \to 0^+} \frac{n}{2\pi} \int_{\pi/2}^{\alpha(r)} d\theta = \frac{n}{2\pi}\left(\frac{\pi}{6} - \frac{\pi}{2}\right) = \frac{-n}{6}.$$



Similarly, if

$$f(z) = (z-i)^m g(z)$$

$(m \in \mathbb{Z},\ g(i) \neq 0)$,

then

$$\frac{1}{2\pi i} \int_{B^{(r)}} \frac{f'(z)}{f(z)}\, dz = \frac{-m}{2}$$

Combining, we get

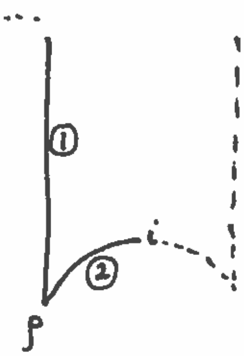
$$Z - P = \ell - \frac{n}{3} - \frac{m}{2} \quad , \text{ where}$$

$$f(z) = \sum_{-\ell}^{\infty} a_k e^{2\pi i k z} \quad (\ell \in \mathbb{Z}) \quad ; \quad f(z) = (z-\rho)^n g(z) \quad \begin{matrix}(g(\rho)\neq 0)\\ n\in\mathbb{Z}\end{matrix}$$

$$= (z-i)^m h(z) \quad \begin{matrix}(h(i)\neq 0)\\ m\in\mathbb{Z}\end{matrix}$$

Z = number of zeroes of f on $\cancel{\{-\tfrac{1}{2} \leq \mathrm{Re} \leq \tfrac{1}{2}\}}$

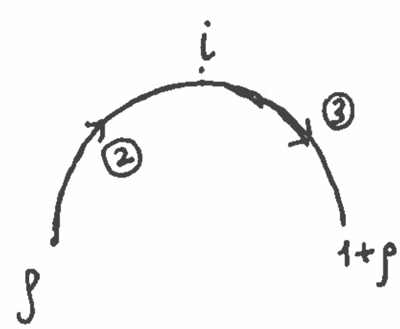
P = " " poles of f on $R \cup$ 1 and 2 $\setminus \{\rho, i\}$



and the theorem follows. $\square$

§5. Proof of Theorem §3. – Now let $f : \mathbb{H} \to \mathbb{C}$ be an entire modular form of weight k. The same calculation as above carries through, except integrals over 2 and 3 do not cancel, and we are left with

$$Z + N(i\infty) + \frac{1}{2} N(i) + \frac{1}{3} N(\rho) = \frac{1}{2\pi i} \int_{2+3} \frac{f'(z)}{f(z)}\, dz$$



$u = S(z)$ change of variables gives

$\left(S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$ $\qquad f(S(z)) = z^k f(z)$

$$\Rightarrow \quad \frac{f'(S(z))\, S'(z)}{f(S(z))} = \frac{k}{z} + \frac{f'(z)}{f(z)}$$

Hence $$\int_\gamma \frac{f'(S(z))}{f(S(z))} S'(z)\, dz = \int_{S(\gamma)} \frac{f'(u)}{f(u)}\, du = \int_\gamma \left(\frac{k}{z} + \frac{f'(z)}{f(z)} \right) dz$$

Take γ to be the arc connecting ρ to i : ②

then $S(\gamma)$ is the arc joining $1+\rho$ to i , ‑③

$$\Rightarrow \int_{②} \frac{f'(z)}{f(z)}\, dz + \int_{②} \frac{k}{z}\, dz = -\int_{③} \frac{f'(u)}{f(u)}\, du$$

So, $$Z + N(i\infty) + \frac{1}{2} N(i) + \frac{1}{3} N(\rho) = \frac{-k}{2\pi i} \int_{\rho \to i} \frac{dz}{z}$$

$$= \frac{-k}{2\pi i} \left(\frac{\pi}{2} i - \frac{2\pi}{3} i \right) = \frac{k}{12}. \qquad \square$$

§6. Corollaries.-

(1) $g_2(\rho) = 0$ with multiplicity 1. g_2 has no other zeroes in $\overline{R}$.

Proof. Since g_2 has weight 4, we have $g_2\left(\frac{-1}{\tau}\right) = \tau^4 g_2(\tau)$

Taking $\tau = \rho$ gives $g_2(-\rho^2) = \rho\, g_2(\rho)$ $\quad (\rho^3 = 1)$

But $g_2(-\rho^2) = g_2(1+\rho) = g_2(\rho)$. Hence $g_2(\rho) = 0$.

Now, from Thm §3, for $g_2(\tau)$, we have $k = 4$ and $N(\rho) \geq 1$:

$$4 = 4\underset{N(\rho)}{\underset{\uparrow}{(1)}} + 12N + 6N(i) + 12N(i\infty)$$ and hence $N = N(i) = N(i\infty) = 0$ and $N(\rho) = 1$ □

(2) $g_3(i) = 0$ with mult 1. g_3 has no other zeroes in $\overline{R}$.

Proof. Again g_3 has weight 6, which gives $g_3\left(\frac{-1}{\tau}\right) = \tau^6 g_3(\tau)$.

Take $\tau = i$ to get $g_3(i) = -g_3(i) \Rightarrow g_3(i) = 0$.

Thm §3 $\Rightarrow$ $6 = 6 \cdot \underbrace{N(i)}_{\geq 1} + \cdots$ so $N(i) = 1$ and the rest are 0. □

(3) $j(\rho) = 0$ with mult 3 ; $j(i) = 1$ with mult. 2.

(proof left as an easy exercise.)