Lecture 21

§1. Hecke operators.- Recall, for each $k \in \mathbb{Z}$ we defined

$$M_k = \left\{ f : \mathbb{H} \to \mathbb{C} \ ; \ \text{hol. at } i\infty \text{ s.t. } f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \ \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ \tau \in \mathbb{H} \right\}$$

weight k- entire modular forms.

$$S_k = \{ f \in M_k : f(i\infty) = 0 \}.$$

<u>Definition.</u> For $n = 1, 2, \ldots$ and $k \in \mathbb{Z}$, define

$$\boxed{(T(n) f)(\tau) = n^{k-1} \sum_{d \mid n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{n\tau + bd}{d^2}\right)} \quad \forall f \in M_k.$$

Lemma.- If $f = \sum_{m=0}^{\infty} c_m e^{2\pi i m\tau}$, then $(T(n)f)(\tau) = \sum_{m=0}^{\infty} \gamma_m e^{2\pi i m\tau}$

where $$\gamma_m = \sum_{d \mid \gcd(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$

Proof. - $$(T(n)f)(\tau) = n^{k-1} \sum_{d \mid n} d^{-k} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} c_m e^{2\pi i m\frac{(n\tau + bd)}{d^2}}$$

$$= \sum_{m=0}^{\infty} \sum_{d \mid n} \left(\frac{n}{d}\right)^{k-1} c_m e^{2\pi i \frac{mn\tau}{d^2}} \underbrace{\left(\frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i m \frac{b}{d}} \right)}_{= \begin{cases} 1 & \text{if } d \mid m \\ 0 & \text{otherwise} \end{cases}}$$

$$\Rightarrow \quad (T(n) f)(\tau) = \sum_{q=0}^{\infty} \sum_{d/n} \left(\frac{n}{d}\right)^{k-1} c_{qd} \, e^{2\pi i q n \tau / d}$$

Change $a = n/d$ to rewrite this as

$$(T(n) f)(\tau) = \sum_{q=0}^{\infty} \sum_{a/n} a^{k-1} c_{q\frac{n}{a}} \, e^{2\pi i q a \tau}$$

Coefficient of $e^{2\pi i m \tau}$ in the expression above is $(qa = m$

$$\gamma_m = \sum_{\substack{a|m \\ a|n}} a^{k-1} c_{\frac{mn}{a^2}} \quad \text{as claimed.} \qquad \square$$

§2. Hecke operators as averaging over orbit space $SL_2(\mathbb{Z}) \backslash \Gamma(n)$.

Let $\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) \; : \; ad - bc = n \right\}$.

<u>Prop.</u> - $$SL_2(\mathbb{Z}) \backslash \Gamma(n) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \begin{array}{l} ad = n \; ; \; a, d > 0 \\ 0 \leq b \leq d-1 \end{array} \right\} =: I(n)$$

Proof. - Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(n)$, choose z, w coprime so that $za + wc = 0$.

Take $x, y \in \mathbb{Z}$ s.t. $xw - yz = 1$. Then $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL_2(\mathbb{Z})$

and $$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$$

$$a_1 d_1 = \det \begin{pmatrix} x & y \\ z & w \end{pmatrix} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = n.$$

Using $\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 + \ell d_1 \\ 0 & d_1 \end{pmatrix}$ we can make sure that $0 \leq b_1 < d_1$ as claimed.

It remains to see that $V \cdot \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$ where

$a_1 d_1 = a_2 d_2 = n$, $0 \leq b_j < d_j$ $(j = 1, 2)$, implies $V = \mathrm{Id}$. This easy check is left as an exercise. $\square$

Hence $T(n)$ can be rewritten as

$$\left(T(n) f\right)(\tau) = n^{k-1} \sum_{\substack{a \geq 1,\ ad = n \\ 0 \leq b < d}} d^{-k} f\left(\frac{a\tau + b}{0\tau + d}\right)$$

$$= \frac{1}{n} \sum_{A \in I(n)} a^k f(A\tau)$$

§3. Theorem.- For $f \in M_k$ (resp. S_k), $T(n) f \in M_k$ (resp. S_k).

Proof.- The proof of this theorem rests on the following easy to verify claim:

If $\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$, where

$\begin{pmatrix} a_j & b_j \\ 0 & d_j \end{pmatrix} \in I(n)$ and $\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} \in SL_2(\mathbb{Z})$, then

$$a_1\left(\gamma_2\ \frac{a_2\tau+b_2}{d_2}+\delta_2\right) = a_2\ (\gamma_1\tau+\delta_1).$$

Given this, let us prove the theorem. Let $M_1 = \begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} \in SL_2(\mathbb{Z})$ be fixed. Then

$$\left(T(n)f\right)(M_1\tau) = \frac{1}{n}\sum_{A_1\in I(n)} a_1^k\ f(A_1M_1\tau)$$

$$= \frac{1}{n}\sum_{A_1\in I(n)} a_1^k\ f(M_2 A_2\tau)$$ where $A_1M_1\in\Gamma(n)$ belongs to coset of $A_2\in I(n)$.

Note $A_1\mapsto A_2$ is a bijection on $I(n)$.

$$= \frac{1}{n}\sum_{A_2} \underbrace{a_1^k\ (\gamma_2 A_2\tau+\delta_2)^k}_{=\,a_2^k(\gamma_1\tau+\delta_1)^k} f(A_2\tau)$$

$$= (\gamma_1\tau+\delta_1)^k\cdot\left(T(n)f\right)(\tau)$$, i.e. $T(n)f\in M_k$

as claimed.

The assertion about cusp forms follows from Lemma §1 above.

$\square$

§4. Theorem. Assume $\gcd(m,n)=1$. Then $T(m)\circ T(n) = T(mn)$.

Proof. - Let $f \in M_k$. Then, by definition we have

$$\Big(\big(T(m)\circ T(n)\big)\cdot f\Big)(\tau) = \frac{1}{mn} \sum_{\substack{A\in I(m)\\ B\in I(n)}} (a_{11}b_{11})^k f(AB\cdot\tau)$$

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12}+a_{12}b_{22} \\ 0 & a_{22}b_{22} \end{bmatrix}$$

$a_{11}a_{22}=m$ $\qquad$ $b_{11}b_{22}=n$

$0\le a_{12}<a_{22}$ $\qquad$ $0\le b_{12}<b_{22}$

Note: divisors of $m\cdot n$ are uniquely of the form $a_{22}\cdot b_{22}$ where $a_{22}\mid m$ and $b_{22}\mid n$.

Claim. -

~~Let~~ $\cancel{\gcd(a,b)=1}$.

$$\left\{ a_{11}b_{12}+a_{12}b_{22} \;\middle|\; \begin{matrix} 0\le a_{12}<a_{22} \\ 0\le b_{12}<b_{22}\end{matrix} \right\} \pmod{a_{22}b_{22}}$$

is the complete set of residues mod $a_{22}b_{22}$.

(this is because $\gcd(a_{ii}, b_{jj})=1 \quad \forall\ \begin{matrix} i=1,2 \\ j=1,2. \end{matrix}$)

Hence $\big(T(m)\circ T(n)\cdot f\big)(\tau) = \dfrac{1}{mn}\displaystyle\sum_{C\in I(mn)} c_{11}^k f(C\cdot\tau)$

$= \big(T(mn)\cdot f\big)(\tau)$ as claimed.

$\square$

§5. Theorem. - Let p be a prime and $r \geq 1$. Then

$$T(p)\,T(p^r) = T(p^{r+1}) + p^{k-1}\,T(p^{r-1}) \quad \text{on } M_k.$$

Proof. -
$$T(p^r)\cdot f = p^{-r} \sum_{\substack{0 \leq t \leq r \\ 0 \leq b_t < p^t}} p^{(r-t)k} f\left(\frac{p^{r-t}\tau + b_t}{p^t}\right)$$

$$\uparrow \quad I(p^r) = \left\{ \begin{bmatrix} p^{r-t} & b_t \\ 0 & p^t \end{bmatrix} : \begin{matrix} 0 \leq t \leq r \\ 0 \leq b_t < p^t \end{matrix} \right\}$$

and
$$T(p)\cdot g = p^{k-1} g(p\tau) + p^{-1} \sum_{b=0}^{p-1} g\left(\frac{\tau+b}{p}\right) \quad \left(I(p) = \left\{\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} : 0 \leq b < p-1\right\} \cup \left\{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right\} \right)$$

Combining these, we get

$$\left(T(p)\,T(p^r) f\right)(\tau) = p^{k-1-r} \sum_{\substack{0 \leq t \leq r \\ 0 \leq b_t < p^t}} p^{(r-t)k} f\left(\frac{p^{r+1-t}\tau + b_t p}{p^t}\right)$$

$$+ \; p^{-1-r} \sum_{\substack{0 \leq t \leq r \\ 0 \leq b_t < p^t}} p^{(r-t)k} \sum_{b=0}^{p-1} f\left(\frac{p^{r-t}\tau + b_t + b p^t}{p^{t+1}}\right)$$

Claim: The second sum on the R.H.S. + $t=0$ term of the first sum is $\left(T(p^{r+1}) f\right)(\tau)$.

(proof. left as an easy exercise)

Hence, $T(p)\,T(p^r)\,f \;=\; T(p^{r+1})\,f$

$$+\; p^{-r-1} \sum_{\substack{1 \le t \le r \\ 0 \le b_t < p^t}} p^{(r+1-t)k} \underbrace{f\left(\frac{p^{r-t}\tau + b_t}{p^{t-1}}\right)}$$

$b_t = q_t \cdot p^{t-1} + r_t$

$0 \le r_t < p^{t-1}$

$$\Rightarrow \left(T(p)\,T(p^r) - T(p^{r+1})\right)\cdot f$$

gives $f\left(\dfrac{p^{r-t}\tau + r_t}{p^{t-1}}\right)$

$$= p^{k-1}\, p^{-r+1} \sum_{\substack{0 \le s \le r-1 \\ 0 \le r_s < p^s}} p^{(r-1-s)k} \cdot f\left(\frac{p^{r-1-s}\tau + r_s}{p^s}\right)$$

There are p different b_t's which give the same r_t mod p^{t-1}.

$(s = t-1)$ and the theorem follows. $\square$

§6. An easy induction argument shows

$$T(p^r)\,T(p^s) = \sum_{t=0}^{\mathrm{Min}(r,s)} p^{t(k-1)}\, T(p^{r+s-2t})$$

$$= \sum_{d \mid \gcd(p^r, p^s)} d^{k-1}\, T\left(\frac{p^{r+s}}{d^2}\right) \qquad (d = p^t)$$

Hence we get

$$\boxed{T(m)\,T(n) = \sum_{d \mid \gcd(m,n)} d^{k-1}\, T\left(\frac{mn}{d^2}\right)} \quad \text{on } M_k.$$