Lecture 22

§1. Recall - last time we introduced Hecke operators $T(n) \in \mathrm{End}_{\mathbb{C}}(M_k)$ $(n \geq 1,\ k \in \mathbb{Z})$.

$$M_k = \left\{ f : \mathbb{H} \to \mathbb{C} \text{ hol. at } i\infty \text{ and } f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ \tau \in \mathbb{H} \right\}$$

$T(n) \circlearrowright M_k$ is given by:

$$(T(n)\cdot f)(\tau) = \frac{1}{n} \sum_{\substack{ad=n \\ 0 \leq b < d}} a^k f\left(\frac{a\tau+b}{d}\right).$$

We proved the following properties of $\{T(n)\}_{n=1}^{\infty}$:

(1) $T(m)\,T(n) = T(mn)$ if $\gcd(m,n) = 1$

(2) $T(p)\,T(p^r) = T(p^{r+1}) + p^{k-1}\,T(p^{r-1})$ where p is a prime and $r \geq 1$.

Combining (1) and (2), we get

$$\boxed{T(m)\,T(n) = \sum_{d \mid \gcd(m,n)} d^{k-1}\, T(mn/d^2)}$$

hence $\{T(n)\}_{n=1}^{\infty}$ is an infinite family of commuting operators on M_k.

(3) If $f(\tau) = \sum_{m=0}^{\infty} c(m)\, e^{2\pi i m\tau}$, then

$$(T(n)f)(\tau) = \sum_{m=0}^{\infty} \Big(\sum_{d \mid \gcd(m,n)} d^{k-1}\, c\big(\tfrac{mn}{d^2}\big) \Big) e^{2\pi i m\tau}$$

In particular, $T(n)$ maps cusp forms to cusp forms.

Two special cases: $m=0$ $(T(n)f)(\tau)$ has constant term $c(0)\cdot \sigma_{k-1}(n)$

If f is a cusp form, i.e. $c(0)=0$, then $T(n)f$ is again a cusp form with leading term $(m=1)$ $c(n)$.

§2. Theorem.– Let $f \in S_k$ be a cusp form ($f \neq 0$)

$$f(\tau) = \sum_{m=1}^{\infty} c(m)\, e^{2\pi i m\tau}.$$

(a) If $T(n)f = \lambda(n) f$ $\forall\, n \geq 1$ (i.e., f is an eigenvector for $\{T(n)\}_{n=1}^{\infty}$) ; $\lambda(n) \in \mathbb{C}$, then $c(1) \neq 0$.

(b) If $c(1) = 1$ (normalized simultaneous eigenform) ; then f is a simultaneous eigenform for Hecke operators $\{T(n)\}_{n=1}^{\infty}$

$$\Longleftrightarrow \quad c(m)\,c(n) = \sum_{d \mid \gcd(m,n)} d^{k-1}\, c\big(\tfrac{mn}{d^2}\big)$$

Proof. $T(n)f = \lambda(n) f$ $\forall\, n \geq 1$ implies $c(n) = \lambda(n)\cdot c(1)$. $\forall\, n \geq 1$

Assuming f is normalized so as to have $c(1)=1$, then

$\exists\ \lambda(n) \in \mathbb{C}$ s.t. $\quad \Longleftrightarrow \quad$ (coeff. of $e^{2\pi i m\tau}$ on both sides)

$$T(n)f = \lambda(n) f \qquad\qquad \sum_{d \mid \gcd(m,n)} d^{k-1}\, c\left(\frac{mn}{d^2}\right) = c(n)\, c(m).$$

(this implies $\lambda(n) = c(n)$)

$\square$

<u>Cor.</u> $$(2\pi)^{-12} \Delta(\tau) = \sum_{m=1}^{\infty} \tau_R(m)\, e^{2\pi i m\tau} \in S_{12} \text{ and } \dim S_{12} = 1.$$

Hence $\{\tau_R(m)\}_{m=1}^{\infty}$ satisfy multiplicative properties:

$$\tau_R(mn) = \tau_R(m)\, \tau_R(n) \quad \text{if } \gcd(m,n) = 1$$

$$\tau_R(p)\, \tau_R(p^r) = \tau_R(p^{r+1}) + p^{\overset{=11}{12-1}}\, \tau_R(p^{r-1})$$

§3. <u>Theorem.</u> Let $f \in M_{2k}$ and assume $c(0) = f(i\infty) \neq 0$.

Then f is a normalized simultaneous eigenform for the Hecke operators $\Longleftrightarrow$ $$f(\tau) = \frac{(2k-1)!}{2(2\pi i)^{2k}}\, G_{2k}(\tau)$$

$$\left(\text{recall } G_{2k}(\tau) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ \neq (0,0)}} \frac{1}{(m+n\tau)^{2k}}\right)$$

Proof. $T(n)f = \lambda(n) f$ implies $c(0)\, \sigma_{2k-1}(n) = \lambda(n)\, c(0)$

so, $\lambda(n) = \sigma_{2k-1}(n) \quad \forall\, n \geq 1$.

Equating coefficients of $e^{2\pi i\tau}$, we get $c(n) = \lambda(n)$
$= \sigma_{2k-1}(n)$.

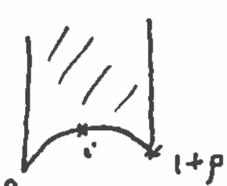
Hence $$f(\tau) = c(0) + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n\tau}.$$

Compare with $$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n\tau}$$

The theorem follows since $\dim\left(M_{2k}/S_{2k}\right) = 1$. $\square$

Remark.- It was shown by Petersson (1939) that the vector space S_{2k} carries a natural Hermitian form, relative to which Hecke operators are self-adjoint. Hence, by their commutativity property, they can be simultaneously diagonalized.

(Petersson inner product) $$(f, g) = \iint_{R_\Gamma} f(\tau)\overline{g(\tau)}\, v^{2k-2}\, du\, dv \qquad (\tau = u + iv)$$

$R =$ 

$e^{2\pi i/3} = \rho$

§4. Modular forms and Dirichlet Series.

$$f(\tau) = c(0) + \sum_{n=1}^{\infty} c(n) e^{2\pi i n\tau} \quad ; \quad \varphi(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

Lemma.- If $\{c(n)\}_{n=1}^{\infty}$ satisfy mult. property

$$c(mn) = \sum_{d \mid \gcd(m,n)} d^{2k-1} c\left(\frac{mn}{d^2}\right), \text{ then } \varphi(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

$$= \prod_{p:\text{ prime}} \frac{1}{1 - c(p)p^{-s} + p^{2k-1-2s}}$$

Proof.- By $c(mn) = c(m)\,c(n)$ if $\gcd(m,n) = 1$, we get

$$\varphi(s) = \prod_{p:\text{ prime}} \left(1 + \sum_{n=1}^{\infty} c(p^n)\, p^{-ns}\right). \quad \text{Now,}$$

$$c(p)\cdot c(p^n) = c(p^{n+1}) + p^{2k-1} c(p^{n-1}) \quad \text{translates to}$$

$$\left(1 - c(p)\, p^{-s} + p^{2k-1} p^{-2s}\right)\left(1 + \sum_{n=1}^{\infty} c(p^n)\, p^{-ns}\right) = 1, \text{ and}$$

the lemma follows. $\square$

<u>Remark.</u> One can estimate $|c_n| = O(n^k)$ if $f = \sum_{n=1}^{\infty} c_n e^{2\pi i n \tau} \in S_{2k}$.

For $\tau_R(n)$, Ramanujan had made a sharper conjecture

$|\tau_R(p)| < 2p^{11/2}$ (proved by Deligne in 1974).

§5. Functional equation for Dirichlet series.

Again let $\varphi(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$. Assume that

$f(\tau) = \sum_{n=1}^{\infty} c(n)\, e^{2\pi i n\tau} \in S_{k}$. Then:

$$(2\pi)^{-s}\,\Gamma(s)\,\varphi(s) = (-1)^{k/2}\,(2\pi)^{s-k}\,\Gamma(k-s)\,\varphi(k-s)$$

Proof.- Start from $\Gamma(s)\,(2n\pi)^{-s} = \int_0^\infty e^{-2\pi n y}\, y^{s-1}\, dy$

So $$(2\pi)^{-s}\,\Gamma(s)\sum_{n=1}^{\infty} c(n)\, n^{-s} = \int_0^\infty \left(\sum_{n=1}^{\infty} c(n)\, e^{-2\pi n y}\right) y^{s-1}\, dy$$

(Ex: Justify $\sum\int = \int\sum$ interchange.)

$$= \int_0^\infty f(iy)\, y^{s-1}\, dy$$

Using $f(iy) = (iy)^{-k} f(i/y)$, we get

$$(2\pi)^{-s}\,\Gamma(s)\,\varphi(s) = \int_1^\infty f(iy)\, y^{s-1}\, dy + \int_0^1 (iy)^{-k} f\left(\frac{i}{y}\right) y^{s-1}\, dy$$

$$= \int_1^\infty f(iy)\, y^{s-1}\, dy + i^{-k}\int_1^\infty f(iw)\, w^{k-s-1}\, dw \qquad (w = y^{-1})$$

i.e., $$(2\pi)^{-s}\,\Gamma(s)\,\varphi(s) = \int_1^\infty f(iy)\left(y^s + (-1)^{k/2}\,y^{k-s}\right)\frac{dy}{y}.$$

$$\left. y^s + (-1)^{k/2}\,y^{k-s} \right|_{s \to k-s} = y^{k-s} + (-1)^{k/2}\,y^s$$
$$= (-1)^{k/2}\left(y^s + (-1)^{k/2}\,y^{k-s}\right)$$

and the functional equation for $\varphi(s)$ follows.

$\square$