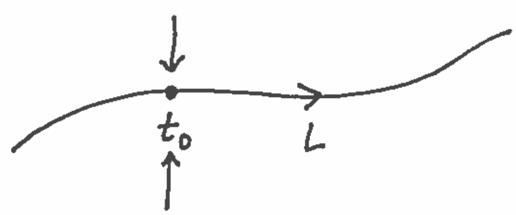
Lecture 23

Our next topic is <u>Riemann-Hilbert factorization problems</u> and their relation with <u>singular integral equations</u>.

§1. An example of a factorization problem is given as follows. Assume $L \subset \mathbb{C}$ is a piecewise smooth path and a $\mathbb{C}$-valued function $\varphi : L \to \mathbb{C}$ is given.

<u>Problem</u>: Find a holomorphic function $\Phi(z)$, $z \in \mathbb{C} \setminus L$ such that, $\forall\, t_0 \in L$; t_0 not an endpoint of L,

$$\lim_{\substack{z \to t_0 \\ \text{from the left}}} \Phi(z) - \lim_{\substack{z \to t_0 \\ \text{from the right}}} \Phi(z) = \varphi(t_0)$$



$$\Phi^{\pm}(t_0) = \lim_{\substack{z \to t_0 \\ \text{from left/right}}} \Phi(z)$$

Many variants of such problems were encountered by Riemann (1851), Hilbert (1904), Poincaré (1910). J. Plemelj (1908) obtained a closed form solution to factorization problem stated above, in what is now known as "Plemelj formulae" – see next section. For general factorization problems and singular integral equations, Plemelj formulae are extremely useful in reducing the problem to linear integral equations of

Fredholm-type (to be discussed later).

§2. <u>Plemelj formulae</u>. - Assume L is a finite (simple) piecewise differentiable path in $\mathbb{C}$, and $\varphi : L \to \mathbb{C}$ satisfies <u>Hölder condition</u>:

i.e. $\exists\, M > 0$ and $0 < \lambda \leq 1$ s.t. $|\varphi(t_1) - \varphi(t_2)| < M \cdot |t_1 - t_2|^{\lambda}$ $\forall\, t_1, t_2 \in L$.

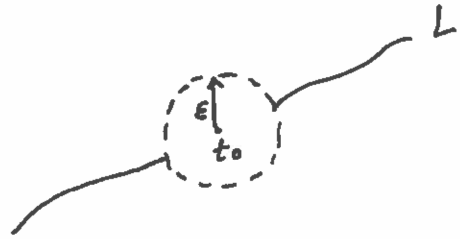
<u>Theorem</u>. - $\Phi(z) := \dfrac{1}{2\pi i} \displaystyle\int_L \frac{\varphi(t)}{t - z}\, dt$ defines a hol. fn. of $z \notin L$.

For any $t_0 \in L$ not an endpoint, the limits $z \to t_0$ from left/right exist and are given by

$$\Phi^{\pm}(t_0) = \pm \frac{1}{2} \varphi(t_0) + \frac{1}{2\pi i} \fint_L \frac{\varphi(t)}{t - t_0}\, dt .$$

Here $\fint$ indicates Cauchy's principal value

$$\fint \frac{\varphi(t)}{t - t_0}\, dt := \lim_{\varepsilon \to 0^+} \int_{L_\varepsilon} \frac{\varphi(t)}{t - t_0}\, dt$$



$$L_\varepsilon = L \setminus \{\tau \in L : |\tau - t_0| < \varepsilon\}$$

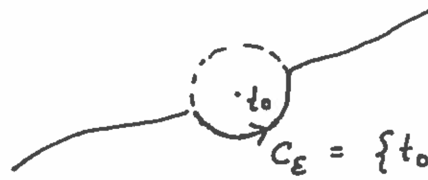
<u>Remark</u>.- An easy, and particularly useful special case is when $\varphi : L \to \mathbb{C}$ is the restriction to L of a hol. fn. defined on an open set containing L - again denoted by $\varphi : \mathcal{U} \to \mathbb{C}$.

In this case, we can use Cauchy's theorem to deform the path of integration and write

$$\Phi^{+}(z) = \frac{1}{2\pi i} \int_{L_\varepsilon^{+}} \frac{\varphi(t)}{t - z}\, dt$$

Hence,

$$\Phi^{+}(t_0) = \frac{1}{2\pi i} \int_{L_\varepsilon^{+}} \frac{\varphi(t)}{t - t_0}\, dt$$



$C_\varepsilon = \{ t_0 + \varepsilon e^{i\theta} : -\pi < \theta < 0 \}$

$$L_\varepsilon^{+} = L_\varepsilon \cup C_\varepsilon$$

Taking the limit, as $\varepsilon \to 0$, we get

$$\Phi^{+}(t_0) = \frac{1}{2\pi i} \fint_{L} \frac{\varphi(t)}{t - t_0}\, dt + \underbrace{\frac{1}{2\pi \cancel{i}} \lim_{\varepsilon \to 0} \int_{-\pi}^{0} \frac{\varphi(t_0 + \varepsilon e^{i\theta})}{\cancel{\varepsilon} e^{i\theta}}\, \cancel{\varepsilon} \cdot \cancel{i} \cdot e^{i\theta}\, d\theta}_{= \frac{1}{2}\varphi(t_0)}$$

Similarly $\Phi^{-}(t_0) = \dfrac{1}{2\pi i} \displaystyle\fint_{L} \frac{\varphi(t)}{t - t_0}\, dt - \frac{1}{2}\varphi(t_0).$

The statement of the theorem above is usually rewritten in an equivalent form:

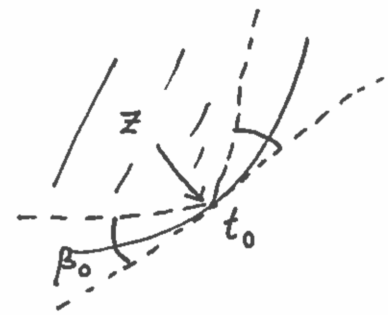
$$\Phi^{+}(t_0) - \Phi^{-}(t_0) = \varphi(t_0)$$

$$\Phi^{+}(t_0) + \Phi^{-}(t_0) = \frac{1}{\pi i} \fint_{L} \frac{\varphi(t)}{t - t_0}\, dt \, .$$

§3. Proof of Theorem §2. is based on the following lemma

Lemma (Muskhelishvili) . .
$$\int_L \frac{\varphi(t)-\varphi(t_0)}{t-z}\,dt \to \int_L \frac{\varphi(t)-\varphi(t_0)}{t-t_0}\,dt$$

as $z \to t_0$ in such a way that the non-obtuse angle between line segment $z t_0$ and tangent to L at t_0 is larger than a fixed $\beta_0 \in (0, \frac{\pi}{2})$.

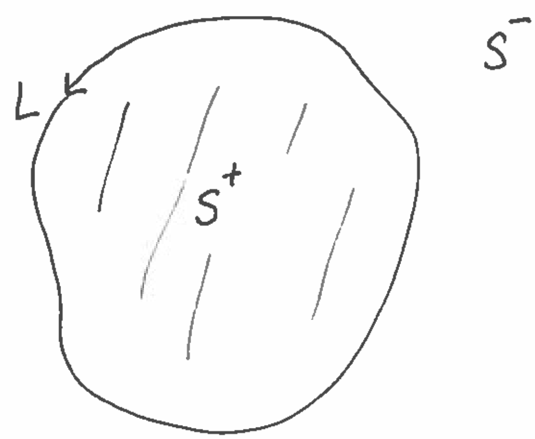


This convergence is uniform in t_0.

We will give a proof of this lemma at the end of these notes. For now, let us assume this and continue with the proof of Thm §2.

First assume that L is a (positively oriented) contour (so closed). Then

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)-\varphi(t_0)}{t-z}\,dt + \frac{\varphi(t_0)}{2\pi i}\int_L \frac{\varphi(t)}{t-z}\,dt$$



The finite (bounded) component of $\mathbb{C} \setminus L$ is to the left of

– positive orientation

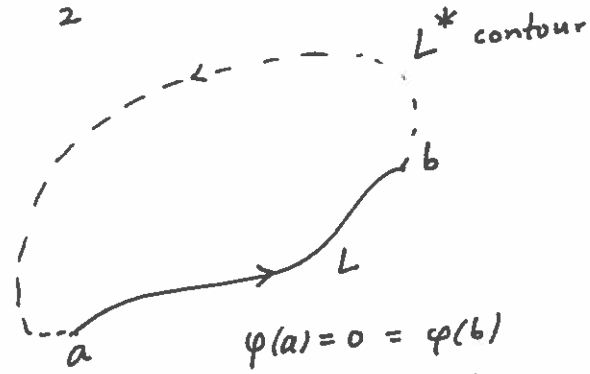
By Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_L \frac{dt}{t-z} = \begin{cases} 1 & \text{if } z \in S^+ \\ 0 & \text{if } z \in S^- \end{cases}$$

So, we get $$\Phi^+(t_0) = \varphi(t_0) + \frac{1}{2\pi i}\int_L \frac{\varphi(t)-\varphi(t_0)}{t-t_0}\,dt$$

$$\Phi^-(t_0) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)-\varphi(t_0)}{t-t_0}\,dt$$

The theorem follows in this case, since for a contour L, as (see Remark §2 above) $$\frac{1}{2\pi i}\int_L \frac{dt}{t-t_0} = \frac{1}{2}.$$



If L is a finite, simple path, say from a to b, and $\varphi(a) = \varphi(b) = 0$, then we can close L and extend φ by 0.

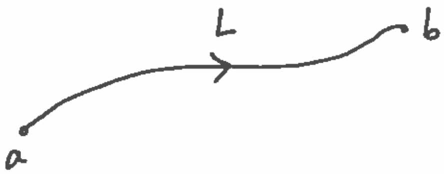
$\varphi(a) = 0 = \varphi(b)$ extended to L^* by 0.

Thus the theorem holds for all $t_0 \in L$ excluding the end points when $\varphi \neq 0$. $\square$

§4. <u>Remarks</u>. - (1) Behaviour of $\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}\,dt$

near the end points a, b of L can be described as follows.

Write $$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(t) - \varphi(a)}{t - z}\, dt$$



$$+ \frac{\varphi(a)}{2\pi i} \underbrace{\int_L \frac{dt}{t-z}}_{\log\left(\frac{z-b}{z-a}\right)}$$

So, near a, we get $\quad \Phi(z) = \dfrac{\varphi(a)}{2\pi i} \log \dfrac{1}{z-a} + \boxed{\Phi_a(z)}$

holomorphic near a

(2) If L is infinite, say e.g. $L = \mathbb{R}$, additional hypothesis on φ need to be imposed –
$$\begin{cases} \varphi(t) \to C \quad \text{as } t \to \pm\infty \text{ and} \\ |\varphi(t) - C| < \dfrac{M}{t^{\mu}} \qquad (M > 0,\ \mu > 0) \end{cases}$$

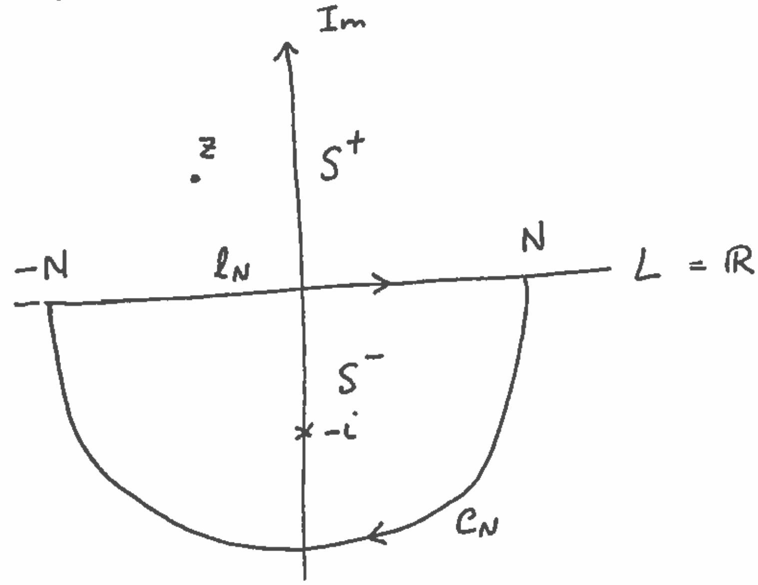
§5. Examples. – (i) $\quad \varphi(t) = t + \frac{1}{t} \quad ; \quad t \in S^1 = C(0,1)$

$$\Phi(z) = \frac{1}{2\pi i} \int \left(t + \frac{1}{t}\right) \frac{dt}{t-z} = \begin{cases} z + \frac{1}{z} - \frac{1}{z} & \text{if } z \in D(0,1) \\ -\frac{1}{z} & \text{if } z \notin \overline{D(0,1)} \end{cases}$$

$C(0,1)$

$\Phi^+(z) = z \quad$ and $\quad \Phi^-(z) = -\frac{1}{z}$.

(ii) $\quad \varphi(t) = \dfrac{1}{1+t^2} \quad ; \quad t \in \mathbb{R}.$



For $z \in S^+$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,\frac{dt}{t-z} = \frac{i}{2(z+i)}$$

can be computed using:

$$\left\{ \begin{array}{l} \displaystyle \frac{1}{2\pi i}\int_{\ell_N + C_N} \frac{dt}{(1+t^2)(t-z)} = \frac{-1}{(-i-z)(-2i)} = \frac{i}{2(z+i)} \\[2ex] \text{and} \quad \displaystyle \left| \int_{C_N} \frac{dt}{(1+t^2)(t-z)} \right| < \frac{\pi \cdot N}{(N^2-1)(N-|z|)} \\ \qquad\qquad\qquad\qquad\qquad\qquad \to 0 \text{ as } N \to \infty. \end{array} \right.$$

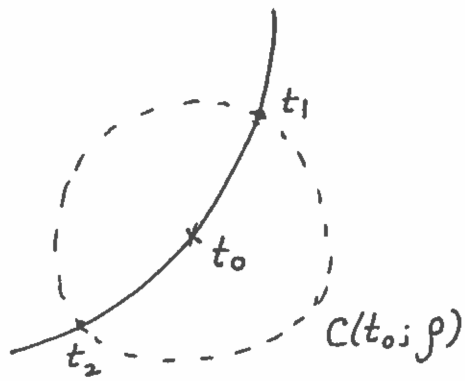
Similarly, for $z \in S^-$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,\frac{dt}{t-z} = \frac{i}{2(z-i)}$$

and $\quad \Phi^+(z) - \Phi^-(z) = \dfrac{i}{2}\left(\dfrac{1}{z+i} - \dfrac{1}{z-i}\right) = \dfrac{1}{z^2+1}.$

§6. Proof of Muskhelishvili's lemma. –

$$\psi(z) := \frac{1}{2\pi i} \int_L \frac{\varphi(t) - \varphi(t_0)}{t - z} dt$$



$\psi(z) - \psi(t_0)$

$$= \frac{h}{2\pi i} \int_L \frac{\varphi(t) - \varphi(t_0)}{(t-t_0)(t-z)} dt \quad (h = z - t_0)$$

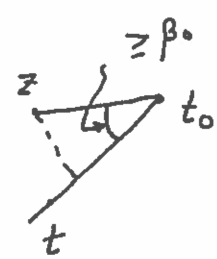
$$= I_1 + I_2, \text{ where } \quad I_1 = \frac{h}{2\pi i} \int_{L_\rho} \frac{\varphi(t) - \varphi(t_0)}{(t-t_0)(t-z)} dt$$

$$I_2 = \frac{h}{2\pi i} \int_{L \setminus L_\rho} \frac{\varphi(t) - \varphi(t_0)}{(t-z)(t-t_0)} dt$$

- For I_1, using the Hölder condition for φ and assumption on $z \to t_0$,

  we get $\quad |I_1| \leq \frac{|h|}{2\pi} C \int_{L_\rho} \frac{r^{\mu-1}}{|t-z|} dr \quad ; \quad |t-z| \geq |h| \sin\beta_0$

$$\leq \frac{C}{2\pi \sin\beta_0} \frac{\rho^\mu}{\mu} \to 0 \quad \text{as } \rho \to 0.$$



- For t on $L \setminus L_\rho$, $\quad |t - t_0| \geq \rho, \quad |t-z| \geq \rho/2$

  Take $\delta \leq \rho/2$ $\quad |I_2| \leq \frac{|h|}{\pi \rho^2} \left(\int_{L \setminus L_\rho} |\varphi(t) - \varphi(t_0)| \, dt \right) \to 0$ as $|h| \to 0$

□