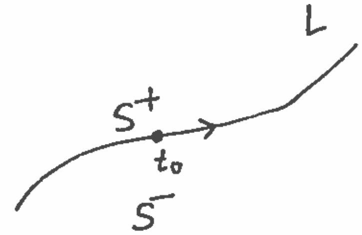
Lecture 24

Recall - In previous lecture we proved Plemelj's formulae



$$\lim_{\substack{z \to t_0 \\ z \in S^\pm}} \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t - z}\, dt$$

$$= \pm \frac{1}{2} \varphi(t_0) + \frac{1}{2\pi i} ⨍_L \frac{\varphi(t)}{t - t_0}\, dt.$$

$\varphi : L \to \mathbb{C}$ satisfying Hölder condition

Here, $⨍_L \frac{\varphi(t)}{t - t_0}\, dt = \lim_{\varepsilon \to 0} \int_{L \setminus D(t_0; \varepsilon)} \frac{\varphi(t)}{t - t_0}\, dt$ is called (Cauchy's) principal value

Equivalent form

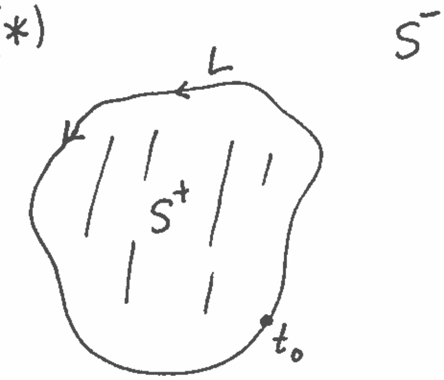
$$\varphi(t_0) = \Phi^+(t_0) - \Phi^-(t_0) \qquad \Phi^\pm(t_0) = \frac{1}{2\pi i} \lim_{\substack{z \to t_0 \\ z \in S^\pm}} \int_L \frac{\varphi(t)}{t - z}\, dt$$

$$\frac{1}{\pi i} ⨍_L \frac{\varphi(t)}{t - t_0}\, dt = \Phi^+(t_0) + \Phi^-(t_0)$$

§1. Application to (scalar, homogeneous) Riemann-Hilbert problem for a smooth contour L.

Assume $g : L \to \mathbb{C}$ Hölder fn s.t. $g(t) \neq 0 \;\forall\, t \in L$ is given. The problem is to find $\Phi^\pm(z)$ $(z \in S^\pm)$ s.t.

$$\boxed{\Phi^+(t_0) = g(t_0)\cdot \Phi^-(t_0) \quad \forall\, t_0 \in L.} \quad (*)$$



We require :

(1) $\Phi^\pm$ are holomorphic on $S^\pm$

and $\Phi^\pm(t) := \lim\limits_{\substack{z\to t\\ z\in S^\pm}} \Phi^\pm(z)$ exist

(2) $\Phi^-(z)$ to have finite order, say m, at ∞. i.e., $\Phi^-(z) = c_m z^m + O(z^{m-1})$ near $z=\infty$.
$(m\in\mathbb{Z})$.

<u>Theorem</u> Let k = index of $g(t)$ around L.

(Assume $0\in S^+$) $\qquad k = \dfrac{1}{2\pi i}\displaystyle\int_L \frac{g'(t)}{g(t)}\,dt$

$$= \frac{1}{2\pi i}\cdot\begin{pmatrix}\text{increment in } \log g(t) \text{ as}\\ t \text{ traverses } L \text{ once}\end{pmatrix} \in \mathbb{Z}.$$

Define

$$G(z) := \frac{1}{2\pi i}\int_L \frac{\log\left(t^{-k} g(t)\right)}{t-z}\,dt \quad \text{and}$$

$$X(z) := \begin{cases} e^{G(z)} & \text{if } z\in S^+ \\ z^{-k} e^{G(z)} & \text{if } z\in S^- \end{cases}$$

Then solution to the scalar, homogeneous R-H problem $(*)$ above, with order m (at most) at infinity is given by $X(z)\cdot \underbrace{P_{m+k}(z)}_{\text{poly of degree } m+k\,\geq\, 0.}$

Proof.- We can take log to make the problem additive, except $\log(g) : L \to \mathbb{C}$ may jump at a start = end point on L by $2\pi i k$ $(k \in \mathbb{Z})$. So, we consider $\cancel{z^{-k}}$ $t^{-k} g(t)$ (assuming $0 \in S^+$ not on L), which has index 0. The equation $(*)$ from the previous page takes the form

$$\Phi^+(t_0) = \left(t^{-k} g(t)\right)\left(t^{k} \Phi^-(t_0)\right)$$

$$\Rightarrow \quad \log \Phi^+(t_0) - \log\left(t^{k} \Phi^-(t_0)\right) = \log\left(t^{-k} g(t)\right)$$

which, by Plemelj formulae, is solved by

$$G(z) = \frac{1}{2\pi i} \int_L \frac{\log\left(t^{-k} g(t)\right)}{t - z}\, dt .$$

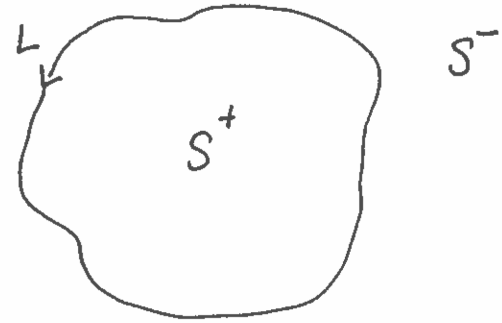
Note that $\quad \Phi^-(z) = z^{-k} \exp\left(G(z)\right)$ where $G(z) \to 0$ as $z \to \infty$

$$\Rightarrow \quad \Phi^-(z) \text{ has order } -k \text{ at } \infty .$$

Thus to get order m, we multiply $\Phi^-(z)$ and $\Phi^+(z)$ by a polynomial of degree $m+k$. $\square$

Note: $g \equiv 1$ in $(*)$ means $\Phi(z)$ extends to an entire function of finite order, pole at ∞ $\leq m$ $(m \in \mathbb{Z}_{\geq 0})$, hence a poly of deg. m.

§2. Inhomogeneous case - L is a contour



Find $\Phi^{\pm} : S^{\pm} \to \mathbb{C}$ hol.

and Φ^- has at most a pole at ∞

s.t.

$$\boxed{\Phi^+(t_0) = g(t_0)\,\Phi^-(t_0) + f(t_0)} \quad -(**)$$

$f, g : L \to \mathbb{C}$ given

$g(t) \neq 0 \ \forall t \in L$

First solve for $X(z)$ s.t. $g(t) = X^+(t)\, X^-(t)^{-1}$ as in the homogeneous case

So, $$X(z) = \begin{cases} \exp\left(\dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{\log(t^{-k} g(t))}{t-z}\,dt\right) & \text{if } z \in S^+ \\ z^{-k} \exp\left(\dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{\log(t^{-k} g(t))}{t-z}\,dt\right) & \text{if } z \in S^- \end{cases}$$

K = index of g.

This turns $(**)$ into

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{f(t)}{X^+(t)} \qquad (X^+(t) \neq 0 \ \forall t \in L)$$

for which one particular solution is given by Plemelj formulae

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi i}\int_L \frac{f(t)}{X^+(t)} \frac{dt}{t-z} := \psi(z).$$

Hence general solution to (**) takes the form

$$X(z)\cdot\Big(\underbrace{P_{m+k}(z)}_{\text{poly. of degree } m+k}+\psi(z)\Big)$$

Note : $X^-(z)\cdot\psi(z) = O(z^{-k-1})$. So, for instance, if we are looking for <u>solutions vanishing at ∞</u>, we have.

(a) $k>0$. In this case, there are k linearly independent solutions of (**) vanishing at ∞ ($m=-1$ above)

(b) $k=0$. There is a unique sol. vanishing at ∞, namely $X(z)\cdot\psi(z)$.

(c) $k<0$. There is a unique solution ($X(z)\cdot\psi(z)$)

$$\Leftrightarrow \quad \int_L \frac{f(t)}{X^+(t)}\, t^{n-1}\, dt = 0 \quad \forall\ n=1,2,\dots,-k.$$

(part (c) $\quad X(z)\cdot\psi(z) \sim O(z^{-k-\ell})$ if $\psi(z)\sim O(z^{-\ell})$

To have $-k-\ell=-1$, we need $\ell = 1-k$. i.e.

$$\psi(z) = \frac{-1}{2\pi i}\int_L \frac{f(t)}{X^+(t)}\left(\sum_{m=0}^{\infty} t^m z^{-m-1}\right) dt$$

should start from z^{k-1}, hence first $-k$ terms should be 0. □)

§3. Cauchy inversion formula.

Again let L be a contour and $f : L \to \mathbb{C}$ Hölder function.

Prop. - If $\varphi(t) := \frac{1}{\pi i} \int_L \frac{f(\tau)}{\tau - t} d\tau$, then

$$f(\tau) = \frac{1}{\pi i} \int_L \frac{\varphi(t)}{t-\tau} dt$$

Proof. - Given f as above, the function $\psi(z) = \frac{1}{2\pi i} \int_L \frac{f(\tau)}{\tau - z} d\tau$

satisfies Plemelj formulae

$$\psi^+(t) + \psi^-(t) = \frac{1}{\pi i} \int_L \frac{f(\tau)}{\tau - t} d\tau = \varphi(t) \qquad - (1)$$

$$\psi^+(t) - \psi^-(t) = f(t). \qquad - (2)$$

But solution to eq^n (1) can also be obtained from $\varphi(t)$ as

$$\frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z} dt, \quad z \in S^+ \qquad , \qquad -\frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z} dt, \quad z \in S^-.$$

By uniqueness, the first is $\psi^+(z)$ and second $\psi^-(z)$

$$\Rightarrow \underset{\text{by eq}^n (2)}{f(t_0) =} \psi^+(t_0) - \psi^-(t_0) = \lim_{\substack{z \to t_0 \\ z \in S^+}} \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z} dt + \lim_{\substack{z \to t_0 \\ z \in S^-}} \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z} dt$$

$$= \frac{1}{\pi i} \int_L \frac{\varphi(t)}{t-t_0} dt \quad \text{as claimed.} \qquad \square$$

Exercise - Show the following (Poincaré - Bertrand formulae)

(1) $$f(t) = \frac{-1}{\pi^2} \int_L \left(\int_L \frac{f(\tau')}{\tau'-\tau}\, d\tau' \right) \frac{d\tau}{\tau - t}.$$

(2) If $f(t_1, t_2)$ is given on $L \times L$ and is Hölder function in each argument, then

$$\int_L \left(\int_L \frac{f(t_1,t_2)}{t_2-t_1}\, dt_2 \right) \frac{dt_1}{t_1-t} - \int_L \left(\int_L \frac{f(t_1,t_2)}{(t_2-t_1)(t_1-t)}\, dt_1 \right) dt_2$$

$$= -\pi^2 f(t,t).$$