Lecture 25

Linear integral equations - Fredholm theory

§1. Integral equations were first introduced by Laplace in 1782 who considered the following integral transformations.

$$f(x) = \int e^{xt} \phi(t)\, dt \qquad g(x) = \int t^{x-1} \phi(t)\, dt$$

Next significant appearance of integral equations was in the works of Fourier (1802)

$$f(x) = \int_{-\infty}^{\infty} \cos(xt)\, \phi(t)\, dt \quad \rightsquigarrow \quad \phi(x) = \frac{2}{\pi} \int_0^{\infty} \cos(ux) f(u)\, du$$

and Abel (1823)

$$f(x) = \int_a^x \frac{u(t)}{(x-t)^\mu}\, dt \qquad 0 < \mu < 1, \quad a \leq x \leq b$$

$$\rightsquigarrow \quad u(z) = \frac{\sin \mu\pi}{\pi} \frac{d}{dz} \int_a^z \frac{f(x)}{(z-x)^{1-\mu}}\, dx.$$

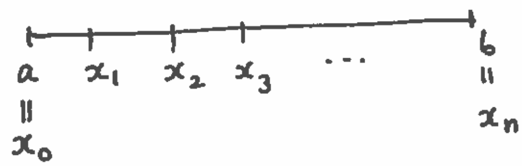
A major step towards a systematic study of such equations was taken by Fredholm in 1903. Fredholm's work has been vastly generalized to "spectral theory of compact operators" - by results and theories of Hilbert spaces.

§2. Fredholm equations. – Given a continuous function $f(x)$ on $a \le x \le b$ and $K(x,y)$, $a \le x, y \le b$, solve for $\phi(x)$ s.t.

$$\boxed{\phi(x) - \lambda \int_a^b K(x,y)\,\phi(y)\,dy = f(x)} \qquad - (*)$$

Fredholm's derivation of $\phi(x)$. –

Consider the subdivision of $[a,b]$ into n equal subintervals



$x_0 = a$, $\quad x_{p+1} - x_p = \delta \left(= \frac{b-a}{n}\right)$

$x_n = b$.

Writing the partial Riemann sums, and taking $(*)$ for $x = x_p$ gives a linear system of equations:

$$\phi(x_p) - \lambda\delta \sum_{q=1}^{n} K(x_p, x_q)\,\phi(x_q) = f(x_p) \qquad 1 \le p \le n$$

By linear algebra, this system is solved by

$$\phi(x_\mu) = \frac{1}{D_n(\lambda)} \cdot \sum_{p=1}^{n} D_n(x_\mu, x_p)\, f(x_p), \quad \text{where}$$

$$D_n(\lambda) = \det\left(Id - \lambda\delta \left(K(x_p, x_q)\right)_{1\leq p,q\leq n} \right)$$

$$= 1 - \lambda\delta \sum_{p=1}^{n} K(x_p, x_p) + \frac{\lambda^2\delta^2}{2!} \sum_{p,q=1}^{n} \begin{vmatrix} K(x_p,x_p) & K(x_p,x_q) \\ K(x_q,x_p) & K(x_q,x_q) \end{vmatrix}$$

$$+ \dots + (-1)^n \frac{\lambda^n\delta^n}{n!} \det\left(K(x_p, x_q)\right)_{1\leq p,q\leq n}.$$

and $D_n(x_\mu, x_\nu)$ is the cofactor in $D_n(\lambda)$ of the term involving $K(x_\nu, x_\mu)$. In the limit, $n\to\infty$ (hence $\delta\to 0$) we find

$$D(\lambda) = 1 - \lambda \int_a^b K(\xi_1, \xi_1)\, d\xi_1 + \frac{\lambda^2}{2!} \int_a^b\!\!\int_a^b \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2$$

$$- \dots$$

and
$$D(x_\mu, x_\nu; \lambda) = \lambda K(x_\mu, x_\nu) - \lambda^2 \int_a^b \begin{vmatrix} K(x_\mu,x_\nu) & K(x_\mu,\xi_1) \\ K(\xi_1,x_\nu) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$+ \frac{\lambda^3}{3!} \int_a^b\!\!\int_a^b \begin{vmatrix} K(x_\mu,x_\nu) & K(x_\mu,\xi_1) & K(x_\mu,\xi_2) \\ K(\xi_1,x_\nu) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,x_\nu) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2 - \dots$$

Hence our solution should be of the form

$$\phi(x) = f(x) + \frac{1}{D(\lambda)} \int_a^b \cancel{K}\, D(x, \xi; \lambda) f(\xi)\, d\xi$$

§3. Given $K(x,y)$ continuous on $[a,b]\times[a,b]$, consider

$$D(\lambda) := 1 + \sum_{n=1}^{\infty} a_n \frac{\lambda^n}{n!} \;;\quad a_n = (-1)^n \int_{[a,b]^n} \Big| \big(K(\xi_i,\xi_j)\big)_{1\le i,j\le n} \Big| \, d\xi_1 \cdots d\xi_n$$

$$D(x,y;\lambda) = \sum_{n=0}^{\infty} \frac{v_n(x,y)\,\lambda^{n+1}}{n!} \;;\quad v_n(\cancel{x},y) = \cancel{(-1)}^n \int\!\!\cancel{\int}_{[a,b]^n} \cancel{K(\xi_i,\xi_j)}_{\cancel{0}\le i,j\le n}$$

where
$$v_n(x,y) = (-1)^n \underbrace{\int_a^b \cdots \int_a^b}_{n\text{-fold}} \begin{vmatrix} K(x,y) & K(x,\xi_1) \;\cdots\; K(x,\xi_n) \\ K(\xi_1,y) & \\ \vdots & \boxed{\;K(\xi_i,\xi_j)_{1\le i,j\le n}\;} \\ K(\xi_n,y) & \end{vmatrix} d\xi_1 \cdots d\xi_n$$

Hadamard's lemma [If $|a_{ij}| < M$ $\forall i,j \in \{1,\dots,n\}$, then $|\det(a_{ij})| \le n^{n/2}\cdot M^n$.

(row sums are bounded as $\displaystyle\sum_{j=1}^{n} \frac{a_{ij}^2}{nM^2} \le 1$, so $\left|\det\left(\dfrac{a_{ij}}{\sqrt{n}\,M}\right)\right| \le 1$.)

$\uparrow$
(Hadamard's lemma)]

easily implies that $D(\lambda)$ and $D(x,y;\lambda)$ are entire functions of λ.

Theorem. - $$D(x,y;\lambda) = \lambda D(\lambda) K(x,y) + \lambda \int_a^b D(x,\xi;\lambda)\, K(\xi,y)\, d\cancel{y}\xi$$

Proof.- Recall $\mathcal{D}(x,y;\lambda) = \sum_{n=0}^{\infty} v_n(x,y)\,\frac{\lambda^{n+1}}{n!}$,

where
$$v_n(x,y) = (-1)^n \underbrace{\int_a^b \cdots \int_a^b}_{n\text{-fold}} \begin{vmatrix} K(x,y) & K(x,\xi_1) \ldots K(x,\xi_n) \\ K(\xi_1,y) & \\ \vdots & \boxed{\begin{matrix} K(\xi_i,\xi_j) \\ 1 \le i,j \le n \end{matrix}} \\ K(\xi_n,y) & \end{vmatrix} d\xi_1 \cdots d\xi_n$$

Expanding the determinant along the first column, we get

$$\text{Integrand of } v_n(x,y) = K(x,y)\cdot \left| \left(K(\xi_i,\xi_j)\right)_{1\le i,j\le n} \right|$$
$$+ \sum_{j=1}^{n} (-1)^j K(\xi_j,y) \begin{vmatrix} K(x,\xi_1) \;\ldots\; K(x,\xi_n) \\ \boxed{\begin{matrix} K(\xi_a,\xi_b) \\ 1\le a,b\le n;\ a\ne j \end{matrix}} \end{vmatrix}$$

renaming variables - we get the same answer upon $\int_{[a,b]^n} (-)\,d\underline{\xi}$.

$$\Rightarrow \mathcal{D}(x,y;\lambda) = \lambda D(\lambda) K(x,y)$$
$$+ \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} \cdot (-1)\cdot n \cdot \underbrace{\int_a^b \cdots \int_a^b}_{n \text{ fold}} \begin{vmatrix} K(x,\xi_0) & \ldots & K(x,\xi_{n-1}) \\ K(\xi_1,\xi) & \ldots & K(\xi_1,\xi_{n-1}) \\ \vdots & & \vdots \\ K(\xi_{n-1},\xi) & \ldots & \end{vmatrix} . K(\xi,y)\; d\xi\, d\xi_1 \cdots d\xi_{n-1}$$

$$= \lambda D(\lambda) K(x,y) + \lambda \int_a^b \mathcal{D}(x,\xi;\lambda) K(\xi,y)\,d\xi \quad \text{as claimed} \quad \square$$

§4. Using the calculation given in the previous section,

the integral eq^n $\quad \phi(\xi) = f(\xi) + \lambda \int_a^b K(\xi, y)\phi(y)\, dy$

gives $\cancel{\int f(\xi) D(\xi}$ (multiply on the left by $D(x, \xi; \lambda)$ and integrate w.r.t. ξ)

$$\int_a^b D(x, \xi; \lambda) f(\xi)\, d\xi = \int_a^b D(x, \xi; \lambda)\phi(\xi)\, d\xi$$

$$- \lambda \int_a^b \int_a^b D(x, \xi; \lambda) K(\xi, y)\phi(y)\, dy\, d\xi$$

Substitute $\quad \lambda \int_a^b D(x, \xi) K(\xi, y)\, d\xi = D(x, y) - \lambda D(\lambda) K(x, y)$

to get $\quad \int_a^b D(x, \xi) f(\xi)\, d\xi = \lambda D(\lambda) \int_a^b K(x, y)\phi(y)\, dy$

$$= D(\lambda)\left(\phi(x) - f(x)\right)$$

So, if $D(\lambda) \neq 0$, Fredholm equation has a unique solution

given by $\quad \boxed{\phi(x) = f(x) + \frac{1}{D(\lambda)} \int_a^b D(x, \xi) f(\xi)\, d\xi}$. $\square$

§5. Some Examples.-

(1) Let $K(x,y) = xy$; $[a,b] = [0,1]$. , $f(x) = x$.

$$D(\lambda) = 1 - \frac{1}{3}\lambda \qquad \left(\text{recall: } D(\lambda) = 1 + \sum (-1)^n \frac{a_n \lambda^n}{n!} \right.$$

$$\left. a_n = \int_{[a,b]^n} \det\left(K(\xi_i, \xi_j)\right) d\underline{\xi} \right)$$

e.g. $n = 2$

$$\begin{vmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_2\xi_1 & \xi_2^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \xi_3^2 \end{vmatrix} = 0$$ since all rows are multiple of $(\xi_1 \ \xi_2 \ \xi_3)$.

$$D(x,y;\lambda) = \lambda xy \quad \text{and} \quad \phi(x) = \frac{3x}{3-\lambda}.$$

(2) $K(x,y) = xy + y^2 \rightsquigarrow D(\lambda) = 1 - \frac{2}{3}\lambda - \frac{1}{72}\lambda^2$

$$D(x,y;\lambda) = \lambda(xy + y^2) + \lambda^2\left(\frac{1}{2}xy^2 - \frac{1}{3}xy - \frac{1}{3}y^2 + \frac{1}{4}y\right)$$

(3) Exercise - from the proof of Thm §3 above, show that

$$\int_a^b D(\xi,\xi;\lambda)\, d\xi = -\lambda \frac{d}{d\lambda}\left(D(\lambda)\right)$$

(4) If $K(x,y) = f_1(x) f_2(y)$ and $A = \int_a^b f_1(x) f_2(x)\, dx$, then

$$D(\lambda) = 1 - A\lambda \quad \text{and} \quad D(x,y;\lambda) = \lambda f_1(x) f_2(y).$$

and soln. to Fredholm eq^n $\phi(x) - \lambda \int_a^b K(x,y)\phi(y)\,dy = f(x)$

is given by $$f(x) + \frac{\lambda f_1(x)}{1 - A\lambda} \int_a^b f(\xi) f_2(\xi)\, d\xi .$$

Ex. - More generally if $K(x,y) = \sum_{\ell=1}^{m} f_\ell(x)\, g_\ell(y)$, then $D(\lambda)$ and $D(x,y;\lambda)$ are polynomials in λ.

§6. If $D(\lambda)$ has a zero at $\lambda = \lambda_0$ to order m, i.e., $(C_m \neq 0)$ near $\lambda = \lambda_0$,

$$D(\lambda) = C_m (\lambda - \lambda_0)^m + \cdots$$

and $D(x,y;\lambda) = g_\ell(x,y)(\lambda - \lambda_0)^\ell + \cdots$,

then, by $\int_a^b D(\xi,\xi;\lambda)\,d\xi = -\lambda D'(\lambda)$ we see that $m-1 \geq \ell$,

Hence $$D(x,y;\lambda) = \lambda D(\lambda) K(x,y) + \lambda \int_a^b K(x,\xi) D(\xi,y)\, d\xi$$

$$\Rightarrow \quad g_\ell(x,y) = \lambda_0 \int_a^b K(x,\xi)\, g_\ell(\xi,y)\, d\xi .$$

So, for each y, $g_\ell(x,y)$ as a fn. of x, is a soln. to the hgs integral eq^n

$$\psi(x) = \lambda_0 \int_a^b K(x,\xi)\,\psi(\xi)\, d\xi .$$

§7. Summary of results. -

(1) In case of symmetric kernel, $K(x,y) = K(y,x)$, a theorem of Schmidt states that $D(\lambda) = 0$ has at least one root $\lambda = \lambda_0$.

(2) At a root λ_0 of $D(\lambda) = 0$, if we have n independent solutions to $\psi(x) = \lambda_0 \int_a^b K(x,y)\psi(y)\,dy$, then

$$n \leq \lambda_0^2 \int_a^b\int_a^b \big(K(x,y)\big)^2\,dx\,dy$$

. Hence, the dimension of the space of solutions is finite - called mult. of λ_0.

(3) Assume $\lambda_1, \lambda_2, \ldots$ are roots of $D(\lambda) = 0$ (listed acc to mult.)

$\phi_1(x), \phi_2(x), \ldots$ orthonormal family of solns of $\lambda_j \int K(x,y)\,\phi_j(y)\,dy = \phi_j(x)$

$$\left(\int_a^b \phi_k(x)\,\phi_\ell(x)\,dx = \delta_{k\ell} \right)$$

and assuming $\displaystyle\sum_{k=1}^{\infty} \frac{\phi_k(x)\,\phi_k(y)}{\lambda_k}$ converges, we have

$$K(x,y) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\,\phi_k(y)}{\lambda_k}$$

and corresponding Fredholm det. / resolvent are given by $\displaystyle\prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_k}\right)$ (if convergent)

In this case, a series solution to Fredholm's equation is given as follows. Given $f(x)$, let

$$b_n = \int_a^b \phi_n(x) f(x)\, dx .$$

Then $$\phi(x) = \sum_{n=1}^{\infty} \frac{b_n \lambda_n}{\lambda_n - \lambda} \phi_n(x)$$ solves

$$\phi(x) - \lambda \int_a^b K(x,y) \phi(y)\, dy = f(x)$$