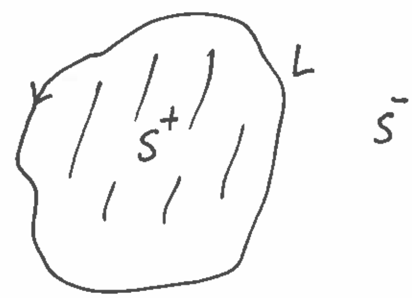
Lecture 26

§0. Recall - we are studying integral equations (singular) and their relation with factorization problems. The crucial result in this setting is Plemelj formulae - stated below

Let L be a contour in the complex plane, (positively oriented) $S^{\pm}$ connected components of $\mathbb{C} \setminus L$ (S^+ is the bounded one - to the left of L)

Given $\varphi : L \to \mathbb{C}$ satisfying Hölder-type condition - we define

$$\Phi(z) := \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z}\, dt$$



Then
$$\lim_{\substack{z \to t_0 \\ z \in S^{\pm}}} \Phi(z) = \pm \frac{1}{2} \varphi(t_0) + \frac{1}{2\pi i} \fint_L \frac{\varphi(t)}{t-t_0}\, dt \qquad \forall\, t_0 \in L$$

Cauchy's principal value.

Thus, if $\Phi^{\pm}(t_0) := \lim_{\substack{z \to t_0 \\ z \in S^{\pm}}} \Phi(z)$, then

$$\boxed{\begin{aligned} \Phi^+(t_0) - \Phi^-(t_0) &= \varphi(t_0) \\ \Phi^+(t_0) + \Phi^-(t_0) &= \frac{1}{\pi i} \fint_L \frac{\varphi(t)}{t-t_0}\, dt \end{aligned}}$$

(Plemelj formulae)

§1. Relation between singular integral equations and Riemann–Hilbert factorization problem.

(a)
$$a(t_0)\,\varphi(t_0) + \frac{b(t_0)}{\pi i}\int_L \frac{\varphi(t)}{t-t_0}\,dt = f(t_0)$$

where $a, b, f : L \to \mathbb{C}$ are given, and we have to solve for φ, is an example of a typical (dominant) singular integral equation.

Let $\Phi(z) := \dfrac{1}{2\pi i}\displaystyle\int_L \frac{\varphi(t)}{t-t_0}\,dt$ as before. By Plemelj formulae, the singular integral equation given above, takes the following form

$$\big(a(t_0)+b(t_0)\big)\,\Phi^+(t_0) - \big(a(t_0)-b(t_0)\big)\,\Phi^-(t_0) = f(t_0)$$

an inhomogeneous R-H factorization problem.

(b) Conversely consider the (homogeneous) R-H factorization problem

$$\Phi^+(t_0) = G(t_0)\,\Phi^-(t_0)$$
$$\Phi^-(z) = \gamma(z) + O(z^{-1}) \quad \text{near } \infty$$

(G and γ are given, and we want to find $\Phi^\pm : S^\pm \to \mathbb{C}$, which extend to cnts. functions on $\overline{S^\pm}$, whose values at $t_0 \in L$ are related by G, as above).

By Cauchy's theorem, we obtain the following equations for $\Phi^\pm$ - if they exist and extend to $\partial S^\pm = L$:

$$\frac{1}{2\pi i}\int_L \frac{\Phi^+(t)\,dt}{t-z} = \begin{cases} \Phi^+(z) & \text{if } z\in S^+ \\ 0 & \text{if } z \in S^- \end{cases}$$

$$\frac{1}{2\pi i}\int_L \frac{\Phi^-(t)}{t-z}\,dt - \gamma(z) = \begin{cases} -\Phi^-(z) & \text{if } z\in S^- \\ 0 & \text{if } z \in S^+ \end{cases}$$

Combine with Plemelj formulae - we get

$$-\frac{1}{2}\Phi^+(t_0) + \frac{1}{2\pi i}\int_L \frac{\Phi^+(t)}{t-t_0}\,dt = 0$$

$$\frac{1}{2}\Phi^-(t_0) + \frac{1}{2\pi i}\int_L \frac{\Phi^-(t)}{t-t_0}\,dt - \gamma(t_0) = 0$$

Hence, $\Phi^+(t_0) = G(t_0)\,\Phi^-(t_0)$ becomes a regular integral eq^n:

$$\Phi^-(t_0) - \frac{1}{2\pi i}\int_L \frac{G(t_0)^{-1}G(t) - I}{t-t_0}\,\Phi^-(t_0)\,dt = \gamma(t_0)$$

Fredholm-type integral equation for Φ^-. Similarly for Φ^+:

$$\Phi^+(t_0) - \frac{1}{2\pi i}\int_L \frac{I - G(t_0)\,G(t)^{-1}}{t-t_0}\,\Phi^+(t)\,dt = G(t_0)\,\gamma(t_0).$$

§2. Fredholm theory. - Recall the main results from the previous lecture.

Given a cnts. fn $K(x,y)$, $x,y \in [a,b]$, we defined

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} a_n \frac{\lambda^n}{n!} \quad \text{and} \quad D(x,y;\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} v_n(x,y) \text{ where}$$

$$v_n(x,y) = (-1)^n \underbrace{\int_a^b \cdots \int_a^b}_{n\text{-fold}} \det \begin{bmatrix} K(x,y) & K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,y) & & & \\ \vdots & & \boxed{\begin{matrix} K(\xi_i,\xi_j) \\ 1 \le i,j \le n \end{matrix}} & \\ K(\xi_n,y) & & & \end{bmatrix} d\xi_1 \cdots d\xi_n$$

and $$a_n = (-1)^n \underbrace{\int_a^b \cdots \int_a^b}_{n\text{-fold}} \det \left[K(\xi_i,\xi_j) \right]_{1 \le i,j \le n} d\xi_1, \ldots d\xi_n$$

Note: $$a_n = -\int_a^b v_{n-1}(\xi,\xi)\, d\xi .$$

Theorem. - (i) $D(\lambda)$ and $D(x,y;\lambda)$ are entire fns. of λ.

(ii) For λ s.t. $D(\lambda) \neq 0$, there is a unique soln. to

$$\phi(x) - \lambda \int_a^b K(x,y)\phi(y)\,dy = f(x)$$

given by $$\phi(x) = f(x) + \frac{1}{D(\lambda)} \int_a^b D(x,y;\lambda) f(y)\,dy .$$

(iii) $$\int_a^b D(\xi,\xi;\lambda)\, d\xi = -\lambda . D'(\lambda)$$

(iv) By col / row expansion of determinant, we get

$$D(x,y;\lambda) = \lambda D(\lambda) K(x,y) + \lambda \int_a^b D(x,\xi;\lambda) K(\xi,y)\, d\xi$$

$$= \lambda D(\lambda) K(x,y) + \lambda \int_a^b K(x,\xi) D(\xi,y;\lambda)\, d\xi$$

(v) If $D(\lambda_0)=0$ to order ~~k~~ m, and $D(x,y;\lambda)$ is expanded in Taylor series near λ_0:

$D(x,y;\lambda) = g_\ell(x,y)(\lambda-\lambda_0)^\ell + \cdots$, then $m \geq \ell+1$ and

$$g_\ell(x,y) = \lambda_0 \int_a^b K(x,\xi)\, g_\ell(\xi,y)\, d\xi$$ solves the hgs. eq^n

$$\psi(x) = \lambda_0 \int_a^b K(x,\xi)\psi(\xi)\, d\xi,$$ (i.e. $g_\ell(x,y_0)$ is a soln. $\forall\, y_0$).

§3. Case of symmetric kernel (assume K is real-valued and $K(x,y) = K(y,x)$)

Theorem. (Schmidt) $D(\lambda)=0$ has a (real) root λ_0.

Proof. Define $K_n(x,y)$, $n \geq 1$, (Liouville's "iterated kernels")

as:
$$K_1(x,y) = K(x,y)$$
$$K_{n+1}(x,y) = \int_a^b K(x,\xi)\, K_n(\xi,y)\, d\xi$$

Then, for $|\lambda| < \dfrac{1}{M(b-a)}$ $\left(M = \max\{|K(x,y)| : x,y \in [a,b]\}\right)$

$$S(x) = f(x) + \lambda \int_a^b K_1(x,y) f(y)\, dy + \lambda^2 \int_a^b K_2(x,y) f(y)\, dy + \cdots$$

converges uniformly and solves

$$S(x) - \lambda \int_a^b K(x,y)\, S(y)\, dy = f(x).$$

Now, let $u_n = \int_a^b K_n(x,x)\, dx$. Comparing the soln found in Theorem §2 above with $S(x) = f(x) + \lambda \int_a^b \rho(x,y;\lambda) f(y)\, dy$

where $\rho(x,y;\lambda) = \sum_{n=0}^{\infty} K_{n+1}(x,y)\, \lambda^n$, we have

$$D(x,y;\lambda) = \lambda D(\lambda)\, \rho(x,y;\lambda).$$ Set $x = y = \xi$ and $\int_a^b$, use Thm §2 (iii) to get $\boxed{\int_a^b \rho(\xi,\xi;\lambda) = -\frac{D'(\lambda)}{D(\lambda)}}$ i.e.,

$$\sum_{n=0}^{\infty} u_{n+1}\, \lambda^n = -\frac{D'(\lambda)}{D(\lambda)}.$$

Next, we show that $u_2, u_4, \ldots$ are all positive and if $v = \frac{u_4}{u_2}$, then $\frac{u_{2n+2}}{u_{2n}} \geq v^n$. The first assertion is clear since K is assumed to be symmetric and $\mathbb{R}$-valued

e.g. $u_2 = \int_a^b K_2(x,x)\, dx = \int_a^b\int_a^b K(x,\xi)\, K(\xi,x)\, d\xi\, dx$

$= \int_a^b\int_a^b |K(x,\xi)|^2\, dx\, d\xi > 0$ and so on.

The second assertion follows from $u_{2n+2}\, u_{2n-2} \geq u_{2n}^2$ which can be proved by using

$$\int_a^b \int_a^b \left(\mu K_{n+1}(x,\xi) + K_{n-1}(x,\xi) \right)^2 dx\, d\xi \geq 0 \quad (\forall \mu \in \mathbb{R})$$

i.e. $\mu^2 u_{2n+2} + 2\mu\, u_{2n} + u_{2n-2} \geq 0$.

In conclusion $\dfrac{D'(\lambda)}{D(\lambda)} = -\sum_{n=0}^{\infty} u_{n+1} \lambda^n$ diverges for $|\lambda^2| \geq \frac{1}{\nu}$. But $D(\lambda)$ is an entire fn. - so is $D'(\lambda)$ - hence $D(\lambda) = 0$ for some λ_0 within or on the circle $|\lambda| = \nu^{-1/2}$. □

§4. Continuing with the set up of §3, assume λ_0 is s.t. $D(\lambda_0) = 0$. Assume that there are n orthonormal solns. to $\boxed{\lambda_0 \int K(x,y)\psi(y)\, dy = \psi(x)}$. (say $\psi_1, \dots, \psi_n$)

We will show that $n \leq \lambda_0^2 \int_a^b \int_a^b K(x,y)^2\, dx\, dy$

hence there are only finitely many solns. to the hys. eq^n

Proof. Let $\Lambda(x,y) := \sum_{m=1}^{n} \underbrace{\left(\int_a^b K(x,\xi)\, \psi_m(\xi)\, d\xi \right)}_{\frac{1}{\lambda_0}\psi_m(x) \text{ from the integral eq}^n.} \psi_m(y)$

Then $$\int_a^b \Lambda(x,y)^2\,dy = \sum_{m=1}^{n}\left(\int_a^b K(x,\xi)\psi_m(\xi)\,d\xi\right)^2$$

$$= \int_a^b K(x,y)\,\Lambda(x,y)\,dy$$

$$\Rightarrow \int_a^b \Lambda(x,y)^2\,dy = * \int_a^b K(x,y)^2 - (K(x,y)-\Lambda(x,y))^2\,dy$$

i.e. $$\int_a^b\left(\sum_{m=1}^{n}\frac{\psi_m(x)\psi_m(y)}{\lambda_0}\right)^2 dy \leq \int_a^b K(x,y)^2\,dy.$$ This means (by orthonormality)

$$\lambda_0^{-2}\sum_{m=1}^{n}\psi_m(x)^2 \leq \int_a^b K(x,y)^2\,dy$$ and the claim follows

upon integrating w.r.t. x. $\square$