# Lecture 27

## §1. Birkhoff Factorization Theorem (1909)

Given $G : S^1 \to GL_n(\mathbb{C})$, there exist $\psi^+ : D(0;1) \to GL_n(\mathbb{C})$

$\psi^- : \mathbb{C} \setminus \overline{D(0,1)} \to GL_n(\mathbb{C})$ and $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ s.t.

$$G(z_0) = \psi^+(z_0) \begin{bmatrix} z^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & z^{\lambda_n} \end{bmatrix} \psi^-(z_0) \qquad \forall z_0 \in S^1.$$

Remarks.- (1) G.D. Birkhoff, Singular points of ordinary linear differential equations (1909)

proved a version of such factorization under different hypothesis on $G$; than the ones imposed by Hilbert and Plemelj - independently. Birkhoff's lemma is presented below.

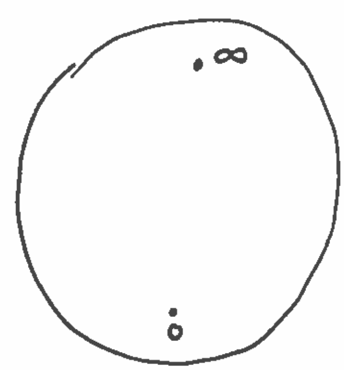
(2) A. Pressley and G Segal - Loop groups is an excellent reference for similar "Birkhoff-style" factorization results where $GL_n(\mathbb{C})$ is replaced by other complex reductive group.

## §2. Holomorphic vector bundles on $\mathbb{P}^1$ ( Grothendieck 1957)

$$\mathbb{P}^1 = \mathcal{U}_0 \cup \mathcal{U}_\infty$$

$$\mathcal{U}_0 = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$$

$$\mathcal{U}_\infty = \mathbb{P}^1 \setminus \{0\} \cong \mathbb{C}$$



Canonical line bundle on $\mathbb{P}^1$ is obtained by gluing

$$\begin{array}{ccc} \mathcal{U}_0 \times \mathbb{C} & & \mathcal{U}_\infty \times \mathbb{C} \\ \downarrow & & \downarrow \\ \mathcal{U}_0 & & \mathcal{U}_\infty \end{array}$$

along the transition map

$$\begin{aligned} \mathcal{U}_0 \cap \mathcal{U}_\infty &\to \mathbb{C}^\times \\ z &\longmapsto z \end{aligned}$$

i.e.

$$L = \mathcal{U}_0 \times \mathbb{C} \sqcup \mathcal{U}_\infty \times \mathbb{C} \Big/ \underset{\text{in } \mathcal{U}_0 \times \mathbb{C}}{(z, \lambda)} \sim \underset{\text{in } \mathcal{U}_\infty \times \mathbb{C}}{(z, z\lambda)} \quad \forall z \in \mathcal{U}_0 \cap \mathcal{U}_\infty = \mathbb{C}^\times$$

$$\downarrow$$

$\mathbb{P}^1$ (projection onto 1st component).

Theorem.- Given a holomorphic vector bundle $E$ on $\mathbb{P}^1$, there exist $a_1, \dots, a_n \in \mathbb{Z}$ ($n =$ rank of $E$) s.t.

$$E \cong L_{a_1} \oplus \dots \oplus L_{a_n}.$$

Here, $\forall k \in \mathbb{Z}$, $L_k := L^{\otimes k}$ if $k \geq 0$ and $(L^*)^{\otimes(-k)}$ if $k < 0$.

Explicitly, the gluing map for $L_k$ is given by $z \mapsto z^k$.

Proof.- $E$ can be trivialized over $\mathcal{U}_0$ and $\mathcal{U}_\infty$, and hence is entirely determined by the gluing data

$$\gamma : \mathcal{U}_0 \cap \mathcal{U}_\infty \to GL_n(\mathbb{C})$$

which is assumed to be a holomorphic function. By the factorization theorem $\gamma(z) = \gamma^+(z) \cdot z^{A} \cdot \gamma^-(z)$

$\gamma^+ : \mathcal{U}_0 \to GL_n(\mathbb{C})$ , $\gamma^- : \mathcal{U}_\infty \to GL_n(\mathbb{C})$ hol.

and $z^a = \begin{bmatrix} z^{a_1} & & 0 \\ & \ddots & \\ 0 & & z^{a_n} \end{bmatrix}$.

Hence, changing the trivialization $\begin{aligned} \mathcal{U}_0 \times \mathbb{C}^n &\to \mathcal{U}_0 \times \mathbb{C}^n \\ (z, v) &\mapsto (z, \gamma^+(z)^{-1} \cdot v) \end{aligned}$

$\begin{aligned} \mathcal{U}_\infty \times \mathbb{C}^n &\to \mathcal{U}_\infty \times \mathbb{C}^n \\ (z, v) &\mapsto (z, \gamma_-(z) \cdot v) \end{aligned}$ , changes the gluing map $\gamma$ to $z^a$

and gives the desired isomorphism $\square$

$$E \cong L_{a_1} \oplus \cdots \oplus L_{a_n}.$$

§3. Argument for $n=1$.

Given $g : \mathbb{C}^\times \to \mathbb{C}^\times$ holomorphic function. Since $g(z) \neq 0$,

$\log(g(z))$ makes sense - as a multivalued fn.

Let $m = \frac{1}{2\pi i} \oint \frac{g'(z)}{g(z)} dz$.

Then $g(z) = z^m \cdot \exp(h(z))$ for a single-valued $h : \mathbb{C}^\times \to \mathbb{C}$.

By Laurent series expansion : $h(z) = \sum_{n \in \mathbb{Z}} h_n z^n$

$$= \underbrace{\sum_{n=0}^{\infty} h_n z^n}_{h^+(z)} + \underbrace{\sum_{n=1}^{\infty} h_{-n} z^{-n}}_{h^-(z)}$$

$\Rightarrow g(z) = z^m \cdot \underbrace{\exp(h_+(z))}_{\substack{\text{hol. near } 0 \\ \text{- infact entire}}} \underbrace{\exp(h_-(z))}_{\substack{\text{hol. near } \infty \\ = 1 \text{ at } z = \infty}}$

§4. Birkhoff's lemma.- Assume given $\Theta(z) : \overset{\times}{D}(\infty; R) \to GL_n(\mathbb{C})$ holomorphic fn.

$\uparrow$ $\{w \in \mathbb{C} : |w| > R\}$

Then $\exists$ $\Lambda(z) : D(\infty; R) \to M_{n\times n}(\mathbb{C})$ (analytic at $\infty$)

$m \in \mathbb{Z}_{\geq 0}$, and $X(z) : \mathbb{C} \to M_{n \times n}(\mathbb{C})$ s.t.

$X(0)$ is invertible and $\boxed{\Theta(z)\cdot\Lambda(z) = z^{-m} X(z)}$.

(same proof gives the transposed factorization $\Lambda_1(z)\,\Theta(z) = z^{-m} X_1(z)$.)

Birkhoff introduces the following operators

$$\sum_{n \in \mathbb{Z}} a_n z^n \xrightarrow{P} \sum_{n=0}^{\infty} a_n z^n$$

$$\xrightarrow{N_\ell} \sum_{n=\ell}^{\infty} a_{-n} z^{-n}$$

and poses the following problem: find $\Gamma$ and $\psi$ s.t.

(*) - $\boxed{\Gamma = P(\Theta^{-1}\psi) \quad \text{and} \quad \psi = z^{-m}\cdot T + N_m(\Theta\cdot\Gamma)}$

(so, $\Theta : \overset{\times}{D}(\infty; R) \to GL_n(\mathbb{C})$, $m \in \mathbb{Z}_{\geq 0}$, and $T \in GL_n(\mathbb{C})$ are given and we want to find $(\Gamma, \psi)$ satisfying (*).)

Claim- If $(\Gamma, \psi)$ is a soln. of (*), then $\Lambda = \cancel{P} N(\Theta^{-1}\psi)$ solves the factorization problem - i.e., $\Theta(z)\Lambda(z) = z^{-m} X(z)$

for $X : \mathbb{C} \to M_{n\times n}(\mathbb{C})$ entire fn.

Proof. - $\theta^{-1}\psi = P(\theta^{-1}\psi) + N(\theta^{-1}\psi)$
$= \Gamma + N(\theta^{-1}\psi) = \Gamma + \Lambda$

And $\psi = T\cdot z^{-m} + N_m(\theta\cdot\Gamma)$. The previous equation gives $\psi = \theta\cdot\Gamma + \theta\cdot\Lambda$, so we get:

$$\theta\cdot\Lambda = T\cdot z^{-m} + \underbrace{N_m(\theta\cdot\Gamma) - \theta\cdot\Gamma}_{\text{of the form } -\sum_{n=-m+1}^{\infty} a_n z^n}$$

$$= z^{-m}\big(T + \underbrace{R(z)}_{\text{hol. in } z,\ =0 \text{ at } z=0}\big)$$

$$= z^{-m}\cdot X(z), \quad X(0) = T \in GL_n(\mathbb{C}).$$ $\square$

The remainder of Birkhoff's argument is to show that for any $T \in GL_n(\mathbb{C})$ and $\theta(z)$, one can find $m \in \mathbb{Z}_{\geq 0}$ s.t. the following "iterated operations" converge to $(\Gamma, \psi)$

$\psi_0 = Tz^{-m}$ $\quad$ $\psi_1 = Tz^{-m} + N_m(\theta\cdot\Gamma_0)$ $\quad \dots \quad$ $\psi_{n+1} = Tz^{-m} + N_m(\theta\cdot\Gamma_n)$

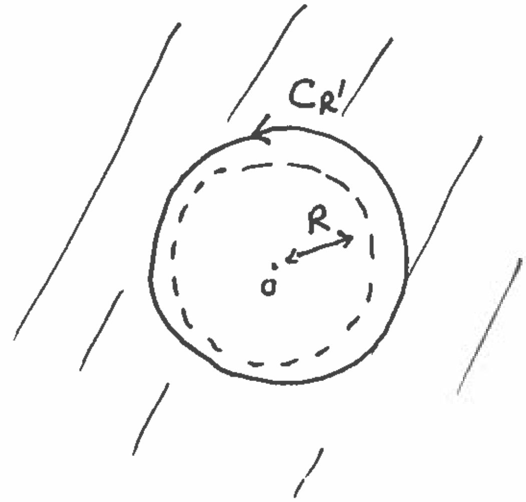
$\Gamma_0 = P(\theta^{-1}\psi_0)$ $\quad$ $\Gamma_1 = P(\theta^{-1}\psi_1)$ $\quad$ $\Gamma_{n+1} = P(\theta^{-1}\psi_{n+1})$

This is nothing but repeated application of $(*)$ for $\psi$:

$$\psi = z^{-m}T + N_m(\theta P(\theta^{-1}\psi))$$ - essentially an integral equation for $\psi$ - see next section.

## §5. Operators $P$ and $N_\ell$ as integrals.-

Recall - Laurent's theorem. -




given $f : D^{\times}(\infty, R) \to \mathbb{C}$,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

where $$a_n = \frac{1}{2\pi i} \int_{C_{R'}} f(z) \cdot z^{-n-1} \, dz .$$

Thus, $$f^{+}(z) = P(f(z)) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_{R'}} f(t)\, t^{-n-1} \, dt \cdot z^n$$

$$= \frac{1}{2\pi i} \int_{C_{R'}} \frac{f(t)}{t-z} \, dt$$

$\left( \frac{1}{t-z} = \sum_{n=0}^{\infty} t^{-n-1} z^n \text{ if } |z| < |t| = R' \right)$

$= -\sum_{n=0}^{\infty} z^{-n-1} t^n \text{ if } |z| > |t| = R'$

and $$N_\ell(f(z)) = \cancel{\sum_{n=-\ell} a_n z^n} \; \sum_{n=\ell-1}^{\infty} a_{-n-1} z^{-n-1}$$

$$= \frac{1}{2\pi i} \int_{C} f(t) \cdot \sum_{n=\ell-1}^{\infty} t^n z^{-n-1} \, dt = \frac{-1}{2\pi i} \int_{C} \frac{f(t)}{t-z} \, dt$$

if $\ell = 1$

$|z| > |t| = R'$.