Lecture 28

§1. Riemann Surfaces. – A <u>Riemann Surface M</u> is a Hausdorff topological space which admits an open cover $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha\in A}$, where

$$M = \bigcup_{\alpha\in A} \mathcal{U}_\alpha \quad ; \quad \varphi_\alpha : \mathcal{U}_\alpha \xrightarrow[\text{homeo.}]{\sim} D_\alpha \subset \mathbb{C} \quad \text{s.t.}$$

an open disc

$$\forall \alpha,\beta\in A, \quad \varphi_\beta\circ\varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha\cap\mathcal{U}_\beta) \to \varphi_\beta(\mathcal{U}_\alpha\cap\mathcal{U}_\beta)$$

is holomorphic.

Remarks.– (1) A theorem of Rado states that a Riemann surface always admits a countable chart – i.e., without loss of generality, we may assume that the indexing set A is finite (if M is compact) or countable.

(2) $\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\sim} D_\alpha$ is often called local coordinate / local chart.

Here, D_α being a disc is not essential. We can take it to be any open, simply-connected subset of $\mathbb{C}$. By Riemann Mapping theorem, up to conformal equivalence, there are only two such subsets: $\mathbb{D}$ or $\mathbb{C}$.

<u>Maps between Riemann Surfaces</u>: Let M and N be two Riemann surfaces and $f: M\to N$ a continuous map. We say f is <u>a holomorphic map</u> if for every $p\in M$ and local charts $(\mathcal{U}_\alpha, \varphi_\alpha)$ around p
(V, ψ) around $f(p)$ s.t. $f(\mathcal{U})\subset V$.

$$\psi\circ f\circ\varphi^{-1} : \varphi(\mathcal{U}) \to \psi(V) \text{ is holomorphic.}$$

When $N = \mathbb{C}$, we say f is a holomorphic function.

When $N = \hat{\mathbb{C}}$, we say f is a meromorphic function.

Examples of Riemann Surfaces. – (1) $\mathbb{C}$, $\hat{\mathbb{C}} = \mathbb{P}^1 =$ Riemann sphere.,

$\mathcal{U} \subset \mathbb{C}$ non-empty open set ; are the standard examples of Riemann surfaces.

(2) Historically, analytic continuation of locally defined functions were the main motivating examples of Riemann surfaces. These examples are best discussed after introducing sheaf of germ of holomorphic functions – so we postpone it for now.

(3) Quotients under group actions. – Assume M is a Riemann surface and $G < Aut(M) = \{ f : M \xrightarrow{\simeq} M \text{ hol. homeomorphism} \}$ be a subgroup s.t. G-action on M is properly discontinuous and free–

i.e. (1) $\forall\, g \in G, \; g \neq e$, $\quad M^g := \{ m \in M : g \cdot m = m \} = \phi$.

(2) $\forall\, K \subset M$ compact, $\{ g \in G : K \cap g(K) \neq \phi \}$ is finite.

Then M/G naturally carries a structure of a Riemann surface so that the natural projection $\pi : M \to M/G$ is holomorphic.

Note : $\pi : M \to M/G$ is a covering map with fiber $\cong G$.

We can relax condition of "free-ness" from $G \circlearrowright M$ and allow points with finite stabilizer groups. In that case, $\pi : M \to M/G$ will be a "ramified" or branched cover.

§2. Local properties of holomorphic maps.-

Recall the inverse function theorem and open mapping theorems from complex analysis.-

Let $\Omega \subset \mathbb{C}$ be an open and connected set, $f : \Omega \to \mathbb{C}$ a holomorphic, non-constant function. Let $z_0 \in \Omega$.

Then $\exists\ \rho_1, \rho_2 > 0$ s.t.

$$f : D(z_0; \rho_1) \to D(f(z_0); \rho_2) \quad \text{is } N\text{-to-}1$$

(here, $N = \text{smallest } \{k \geq 1 \text{ s.t. } f^{(k)}(z_0) \neq 0\}$)

In particular, f is an open map - and up to re-parametrizing discs -
$$f(z) = z^N \text{ in a neighbourhood of } z_0.$$

This result automatically gives us the following local characterization of holomorphic maps between Riemann surfaces.

Theorem.- Let M, N be two connected Riemann surfaces and $f : M \to N$ a non-constant holomorphic map. Then,

(i) $\forall\, n \in N$, $f^{-1}(n) \subset M$ is a discrete set.

(ii) f is an open map (i.e. $\forall\, m \in M$, $\exists\, \mathcal{U} \overset{\text{open}}{\subset} M$, $m \in \mathcal{U}$ s.t. $f(\mathcal{U}) \subset N$ is open – or equivalently – $f(\mathcal{U})$ is open $\forall\, \mathcal{U} \underset{\text{open}}{\subset} M$.)

(iii) $\forall\, m \in M$, $\exists$ an open chart $\varphi : \underset{\varphi(m)=0}{\mathcal{U}} \overset{\simeq}{\to} \mathbb{D}$ around m

and an open chart $\psi : \underset{\psi(f(m))=0}{V} \overset{\simeq}{\to} \mathbb{D}$ around $f(m)$

s.t.

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow[\varphi]{\simeq} & \mathbb{D} & \ni & z \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow[\psi]{\simeq} & \mathbb{D} & \ni & z^N \end{array}$$

(N = order of vanishing of f at m
= smallest $\{k \geq 1 : f^{(k)}(m) \neq 0\}$)

This N is often denoted by $\nu_m(f)$ - order of f at m

A <u>branch point</u> of f is any $p \in M$ where $\nu_p(f) > 1$.

§3. Compact Riemann Surfaces and branched covers.

Assume M and N are connected Riemann surfaces, M is compact and $f : M \to N$ is a non-constant holomorphic map. Then:

(a) N is compact and f is surjective.

This is because $f(M)$ is both open (by open mapping theorem) and closed (since M is compact); and N is connected.

(b) There is a unique positive integer, degree of f, denoted by $\deg(f)$ s.t. $\forall\, q \in N$,

$$\sum_{p \in f^{-1}(q)} v_p(f) = \deg(f).$$

Proof.- Let $n \in N$ be arbitrary. As M is compact and $f^{-1}(n) \subset M$ is discrete - it is finite. So, let

$$f^{-1}(n) = \{m_1, \dots, m_r\}.$$

$\forall\, 1 \leq j \leq r$, we have local charts around m_j s.t. $f(z) = z^{d_j}$ near m_j

$$\begin{tikzcd} m_j \in \mathcal{U}_j \arrow[r, "\sim"', "\varphi_j"] \arrow[d, "f"'] & \mathbb{D} \arrow[d] \ni z \arrow[d, mapsto] \\ n \in V_j \arrow[r, "\sim"', "\psi_j"] & \mathbb{D} \ni z^{d_j} \end{tikzcd}$$

. Define $\deg(f;n) = \sum_{j=1}^{r} d_j$.

<u>Note</u>: There is a neighbourhood of $n \in N$, say $V \subset N$, s.t.

$$\forall\, q \in V \setminus \{n\}, \quad |f^{-1}(q)| = \deg(f;n).$$

Hence $n \mapsto \deg(f;n)$ is a locally constant function - since N is connected, it must be constant. $\square$

The existence of meromorphic functions on an abstract Riemann surface is a non-trivial result.

§4. Cor. – There are no non-constant holomorphic functions on a connected, compact Riemann surface.

Remark. – There exist non-constant meromorphic functions on compact Riemann Surfaces. – i.e. every compact Riemann surface is a branched cover of $\mathbb{P}^1$.

e.g. $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ $\quad z \mapsto p(z)$, $\infty \mapsto \infty$ $\quad$ (p a poly. of deg. n)

is n-to-1 branched cover.

$\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \longrightarrow \mathbb{P}^1 \quad z \longmapsto \wp(z)$ (Weierstrass' $\wp$-function)

is 2-to-1 cover.

In general, if $f : M \to N$ is a degree n branched cover

g_M, g_N : genra of M and N

(if a surface X (compact, oriented) admits a triangulation with V vertices, E edges, F faces, then

$\chi(X) = V - E + F$ is even and independent of the chosen triangulation

genus is defined as $2 - 2g_X = \chi(X)$.)

and $\beta := \sum_{p \in M} (\nu_p(f) - 1)$ total branch number

Riemann-Hurwitz formula

$$\boxed{g_M - 1 = n\,(g_N - 1) + \tfrac{1}{2}\beta}$$

(idea of proof. – let $\mathcal{B} \subset M$ consist of branch points

$$\mathcal{B} = \{ p \in M \; : \; v_p(f) > 1 \}.$$

let T_N be a triangulation of N s.t. $\{ f(p) : p \in \mathcal{B} \}$ are vertices.

T_M = lift of T_N to M. [# vertices in T_M = (# vertices in T_N)$\cdot n$ $- \beta$
edges/faces in T_M $= n$ (# edges/faces in T_N)]

$$2(1-g_N) = V - E + F$$

$$2(1-g_M) = nV - \beta - nE + nF$$

$$= n(2-g_N) + \beta. \qquad \square\,)$$

e.g. $\mathbb{P}^1 \xrightarrow{z^n} \mathbb{P}^1$ $\quad g = 0 \quad \beta = 2n-2 \quad (\, v_\infty = v_0 = n\,;\; v_\alpha = 1 \;\forall \alpha \in \mathbb{P}^1,\ \alpha \neq 0, \infty)$

$\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \to \mathbb{P}^1$ $\quad g_M = 1, \; g_N = 0 \quad n = 2$ and $\beta = 4$.