Lecture 29

§1. Historical remarks on genus and Riemann-Hurwitz formula.-

(a) The term 'genus' γένος is usually attributed to Aristotle

It was used to describe a topological invariant of (compact, oriented) surfaces discovered by Riemann (1851), Jordan (1867), Clebsch (1867) and Poincaré (1895).

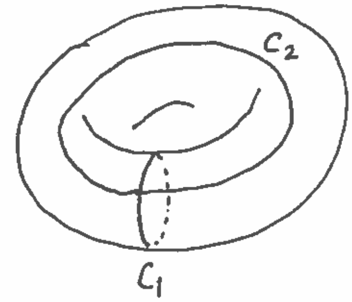
(b) Riemann's defn. Let X be a compact Riemann surface.

Let $N = \mathrm{Max}\left\{ r \;\middle|\; \exists\, r \text{ loops on } X,\ C_1, \dots, C_r \text{ s.t. } X \setminus \bigcup_{j=1}^{r} C_j \text{ is connected} \right\}$

Then N is even and $g_X := \frac{1}{2} N$.

e.g. $X = \mathbb{P}^1$, then $N = 0$ (any loop on the surface of a sphere disconnects the sphere - Jordan Curve Theorem).

$X = \mathbb{C}/_{\mathbb{Z} + \tau\mathbb{Z}}$, $N = 2$



$X \setminus (C_1 \cup C_2) \cong \mathbb{D}$.

It was proved by Jordan (1867) that g_X is a topological invariant of surfaces.

(c) Poincaré (1895) found the following relation between genus g_X and Euler-Poincaré characteristic $\chi(X)$.

Poincaré generalized Euler's formula (1758) $V+F = 2+E$

(for a convex polyhedron – i.e., triangulation of sphere – with V vertices, E edges and F faces, $V+F = 2+E$.)

to $\boxed{V-E+F = 2-2g_X}$ where V, E, F as before are number of vertices, edges and faces of a triangulation of X. Note: such a triangulation defines a chain complex

$$0 \to \mathbb{Z}^F \to \mathbb{Z}^E \to \mathbb{Z}^V \to 0$$

whose Euler-Poincaré characteristic number is defined as the alternating sum of ranks $\chi = V-E+F$.

In terms of (simplicial) homology, we have

$$H_0(X, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(X; \mathbb{Z}) \quad \text{and}$$

$$H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g_X}.$$

That rk $H_1(X,\mathbb{Z})$ must necessarily be even was explained by Poincaré using what is known as "Poincaré duality theorem. Namely, the intersection pairing on $H_1(X; \mathbb{Z})$ is a non-degenerate, skew-symmetric form

$$H_1(X, \mathbb{Z}) \times H_1(X, \mathbb{Z}) \to \mathbb{Z}$$

hence $H_1(X, \mathbb{Z})$ is of even rank.

(d) Riemann-Hurwitz formula. was stated by Riemann (1857) and proved by Hurwitz, It relates degree of a holomorphic map $f : M \to N$ between two compact, connected Riemann surfaces; genra g_M, g_N of M and N respectively; and the total branching number $\beta(f) := \sum_{p \in M} (\nu_p(f) - 1)$:

$$\boxed{2 - 2g_M = \deg(f)\,(2 - 2g_N) - \beta(f)}$$

(recall : if f is n-to-1 near $p \in M$, then $\nu_p(f) := n$)

§2. Sheaf of germs of holomorphic functions.- Let Σ be a Riemann Surface. For any $p \in \Sigma$, let $\mathcal{O}_p(\Sigma)$ (or just $\mathcal{O}_p$ if Σ is clear from the context) be defined as

$$\mathcal{O}_p := \frac{\left\{ (f, \mathcal{U}) : \begin{array}{l} f : \mathcal{U} \to \mathbb{C} \text{ holomorphic} \\ p \in \mathcal{U} \subset \Sigma \text{ open set} \end{array} \right\}}{(f, \mathcal{U}) \sim (g, V) \text{ if } f = g \text{ on } \mathcal{U} \cap V}$$

if we choose a local coordinate nbd. of p - say $p \in \mathcal{U} \underset{\psi}{\overset{\simeq}{\to}} \mathbb{D}$, $\psi(p) = 0$

we can identify $\mathcal{O}_p \cong \mathbb{C}\{z\} \subset \mathbb{C}[[z]]$

$\mathbb{C}\{z\}$: power series with positive radius of convergence.

Introduce a topology on $|\mathcal{O}| := \bigcup_{p \in \Sigma} \mathcal{O}_p$ so that $\begin{array}{c} |\mathcal{O}| \\ \pi \downarrow \\ \Sigma \end{array}$

π is continuous; as follows

(in fact local homeomorphism)

Let $\xi = (p, f) \in |\mathcal{O}|$; i.e. $f \in \mathcal{O}_p$. So, $\exists$ an open set $p \in \mathcal{U} \underset{\text{open}}{\subset} \Sigma$ and $f : \mathcal{U} \to \mathbb{C}$ is holomorphic. Define

$\mathcal{N}(\xi)$ (neighbourhood of ξ) $:= \{ (q, [f]_q) : q \in \mathcal{U}$ and $[f]_q$ denotes equivalence class in $\mathcal{O}_q$ containing $f \}$

The topology on $|\mathcal{O}|$ is generated by basic open sets $\{N(\xi)\}_{\xi \in |\mathcal{O}|}$

Easy check.- $\pi : |\mathcal{O}| \to \Sigma$ is local homeomorphism. Hence $|\mathcal{O}|$ carries a natural structure of a Riemann surface s.t. π is holomorphic.

Remarks- (1) π is not a covering map.

(2) $|\mathcal{O}|$ is the total space of sheaf of holomorphic functions

$$\mathcal{O} : \mathcal{U} \underset{\text{open}}{\subset} \mathbb{C} \longmapsto \mathcal{O}(\mathcal{U}) = \{ f : \mathcal{U} \to \mathbb{C} \text{ holomorphic} \}$$

$\updownarrow$ canonically identified with

$$\{ s : \mathcal{U} \to |\mathcal{O}| \text{ hol. s.t. } \pi \circ s = Id_{\mathcal{U}} \}$$

(3) There is a canonical holomorphic function - namely evaluation - on $|\mathcal{O}|$

$$ev : |\mathcal{O}| \to \mathbb{C} .$$
$$(p, f) \longmapsto f(p)$$

§3. Analytic continuation.

Let $\gamma : [0,1] \to \Sigma$ be a path and $f \in \mathcal{O}_{\gamma(0)}$.

<u>Analytic continuation of f along γ</u> is, by definition, lift of γ to $|\mathcal{O}|$ (assuming such exists), evaluated at 1.

$$\left(\tilde{\gamma}_f : [0,1] \to |\mathcal{O}| \text{ s.t. } \quad \pi \circ \tilde{\gamma}_f = \gamma \quad \text{and } \tilde{\gamma}_f(0) = (\gamma(0), f) \,. \right)$$

$$\begin{tikzcd} & {|\mathcal{O}|} \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\gamma}_f"] \arrow[r, "\gamma"'] & \Sigma \end{tikzcd} \qquad \tilde{\gamma}(0) = (\gamma(0), f) \in \mathcal{O}_{\gamma(0)}$$

$$An_\gamma(f) : \mathcal{O}_{\gamma(0)} \dashrightarrow \mathcal{O}_{\gamma(1)}$$

Riemann Surface associated to local germ $f \in \mathcal{O}_p$ $(p \in \Sigma)$ is by definition connected component of $(p, f) \in |\mathcal{O}|$;

Remark. – (1) Since $|\mathcal{O}|$ admits a holomorphic fn., so does $C(f;p) \subset |\mathcal{O}|$, and therefore is not compact.

(2) Assuming $f \in \mathcal{O}_p(\Sigma)$ is s.t. $An_\gamma(f)$ exists for every path $\gamma : [0,1] \to \Sigma$. Then $\begin{matrix} C(f;p) \\ \downarrow \\ \Sigma \end{matrix}$ is a covering map.

(3) Next, assume $\exists \{p_1, \ldots, p_n\} \subset \Sigma$ s.t. $An_\gamma(f)$ exists for any $\gamma: [0,1] \to \Sigma \setminus \{p_1, \ldots, p_n\}$.

We say f has algebraic singularity at $p = p_j$ $(1 \leq j \leq n)$ if $\exists N \in \mathbb{Z}_{\geq 1}$: $An_{\gamma^N}(f) = f$, where γ is a counterclockwise loop around p and $ev: \pi^{-1}(\mathcal{U} \setminus \{p\}) \cong \mathbb{D}^\times \to \mathbb{C}$ has removable sing. at 0.

Check. $\pi: C(f,p) \longrightarrow \Sigma \setminus \{p_1, \ldots, p_n\}$ near $p = p_j$ is of the form

$$\begin{aligned} \mathbb{D}^\times &\longrightarrow \mathbb{D}^\times. \\ z &\longmapsto z^k \end{aligned}$$

In many texts, Riemann surface of a local germ is assumed to contain branch points ~~as~~ mapping to an algebraic singularity. With this modification, $\Sigma(f;p) := C(f;p) \cup$ branch points over alg. sing.

$$\Sigma(f;p) \big\downarrow \Sigma$$

is a branched / ramified cover.

e.g. $\Sigma = \mathbb{P}^1$, $f = z^{1/2} = (1 + (z-1))^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (z-1)^n \in \mathcal{O}_1(\mathbb{P}^1)$

admits analytic continuation along all paths $\gamma: [0,1] \to \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^\times$

and over $\mathbb{C}^\times$ we have $C(f;1) \cong \mathbb{C}^\times \xrightarrow{\pi} \mathbb{C}^\times$

$$z \longmapsto z^2$$

Thus $\quad \Sigma(f;1) = \mathbb{P}^1 = \mathbb{C}^\times \cup \{0\} \cup \{\infty\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \qquad \uparrow$

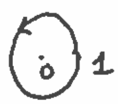
order 2 branch points.

e.g. $\quad f(z) = e^{1/z}$ near $z = 1 \quad \in \mathcal{O}_1(\mathbb{P}^1)$

$$C(f;1) = \mathbb{C}^\times = \mathbb{P}^1 \setminus \{0,\infty\} \xrightarrow[\pi = id]{} \mathbb{C}^\times$$

0 and ∞ are non-algebraic singularities. Even though

$$\mathrm{An}(f) = f \; ; \quad C(f;1) \longrightarrow \mathbb{C} \qquad \text{has essential}$$
$$z \longmapsto e^{1/z} \qquad \text{singularity at } 0.$$



Note :

$$\begin{tikzcd} \Sigma(f;p) \arrow[r, "ev"] \arrow[d, "\pi"'] & \mathbb{C} \cup \{\infty\} = \mathbb{P}^1 \\ \Sigma \supset_{\substack{\text{open} \\ \text{(exists)}}} \mathcal{U} \ni p & \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} (p,f) \in \widetilde{\mathcal{U}} \arrow[d, "\pi", "\cong"'] \arrow[dr, "ev"] & \\ p \in \mathcal{U} \arrow[r, "f"'] & \mathbb{C} \end{tikzcd} \quad \text{Commutes.}$$