§1. Recall that last time we defined, for a Riemann Surface Σ,

<u>sheaf of germs of holomorphic functions.</u> – denoted by $\mathcal{O}(\Sigma)$ –

Total Space $|\mathcal{O}(\Sigma)| = \{ (p,f) : p \in \Sigma, f \in \mathcal{O}_p(\Sigma) \}$ with topology generated by $N(p,f) = \{ (q, [f]_q) : q \in \mathcal{U} \}$ where $\mathcal{U} \subset \Sigma$ open is s.t. $f \in \mathcal{O}_p$ has a representative, still denoted by $\underset{\text{hol.}}{f : \mathcal{U} \to \mathbb{C}}$.

As a sheaf $\mathcal{O}(\Sigma) : \mathcal{U} \longmapsto Hol(\mathcal{U}; \mathbb{C})$ hol fns. on $\mathcal{U}$.

$$\begin{array}{c} |\mathcal{O}(\Sigma)| \\ \pi \downarrow \\ \Sigma \end{array}$$ is a hol. map which is a local iso. ($\forall$ $(p,f) \in |\mathcal{O}(\Sigma)|$ $\exists$ an open neighbourhood $N(p,f) \subset |\mathcal{O}(\Sigma)|$ s.t. $\pi : N(p,f) \to \pi(N(p,f))$ is homeo.)

§2. <u>Ramified and unramified Riemann surfaces associated to a local germ</u>

Given $p \in \Sigma$ and $f \in \mathcal{O}_p$ let $C(p,f)$ = connected component of $(p,f) \in |\mathcal{O}(\Sigma)|$

↑

(unramified R.S. of $f \in \mathcal{O}_p$).

Explicitly, $(q,g) \in C(p,f)$ if there is a path $\tilde{\gamma} : [0,1] \to C(p,f)$ st. $\tilde{\gamma}(0) = (p,f)$, $\tilde{\gamma}(1) = (q,g)$. Let $\gamma = \pi \circ \tilde{\gamma} : [0,1] \to \Sigma$.

γ is then a path joining p and q.

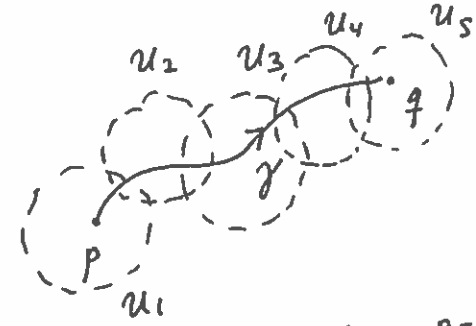
Moreover $\forall\, t \in [0,1]$, $\tilde{\gamma}(t) = (\gamma(t), \xi_{\gamma(t)} \in \mathcal{O}_{\gamma(t)})$

↑ local germ at $\gamma(t)$, say represented by $h_{\gamma(t)} : \mathcal{U}_t \to \mathbb{C}$, $\gamma(t) \in \mathcal{U}_t$ open nhd.

In other words – by defn. of topology on $|\mathcal{O}(\Sigma)|$ – γ can be covered by open sets and one can find hol. fns. on these open sets –

which agree on intersections, and $= f$ at $t=0$.

Summarizing



$(q,g) \in C(p,f)$

$\Leftrightarrow \ \exists \ \gamma : [0,1] \to \Sigma$

$\gamma(0) = p \ , \ \gamma(1) = q$

s.t. γ can be covered $\quad \mathrm{Image}(\gamma) \subset \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ $\left(\text{say } p = \gamma(0) \in \mathcal{U}_1 \atop q = \gamma(1) \in \mathcal{U}_n\right)$

and $\exists$ hol. fns. $\quad f_j : \mathcal{U}_j \to \mathbb{C}$ s.t.

$[f_1] = f \ ; \ [f_n] = g$ and $f_1, \ldots, f_n$ agree wherever their domains intersect.

(Weierstrass' original defn. of analytic continuation).

Ramified Riemann surface $\Sigma(p,f)$ is obtained from $C(p,f)$ by adjoining "algebraic critical points".

Namely, assume for simplicity that there is a discrete set of points $\mathcal{A} \subset \Sigma$ s.t. $f \in \mathcal{O}_p$ admits analytic extn. along any path $\gamma : [0,1] \to \Sigma \setminus \mathcal{A}$. In other words $\mathrm{Image}\left(\begin{array}{c} C(p,f) \\ \pi \downarrow \\ \Sigma \end{array} \right) = \Sigma \setminus \mathcal{A}$.

One can classify points $a \in \mathcal{A}$ – similar to classification of isolated singularities of analytic functions into removable, pole or essential – as follows. Consider a small (punctured) disc around a,

$$\mathbb{D}^\times(a) \subset \Sigma \ \text{ and } \ \pi^{-1}(\mathbb{D}^\times(a)) \subset C(p,f).$$

As π is holomorphic and non-constant – $\pi^{-1}(q) \subset C(p,f)$ is discrete $\forall\, q \in \mathbb{D}^\times(a)$

Case 1. $\forall\, q \in \mathbb{D}^\times(a)$, $\pi^{-1}(q)$ is infinite. – non-algebraic – at least logarithmic singularity at a.

When $\cancel{\mathbb{D}^\times(a) \text{ is}}$ $|\pi^{-1}(q)| = n$ is finite (independent of q – explain why?)

we necessarily have

$$\begin{tikzcd} \pi^{-1}(\mathbb{D}^\times(a)) \arrow[d, "\pi"] \arrow[r, phantom, "\cong"] & \mathbb{D}^\times(0,1) \ni z \arrow[d, mapsto] \\ \mathbb{D}^\times(a) \arrow[r, phantom, "\cong"] & \mathbb{D}^\times(0,1) \ni z^n \end{tikzcd}$$

for some integer n

Case 2.

We say $a \in \Sigma$ is an algebraic singularity of f if

$$\pi^{-1}(\mathbb{D}^\times(a)) \xrightarrow{ev} \mathbb{C}$$ has at most a pole at 0.

Ramified Cover of local germ $f \in \mathcal{O}_p(\Sigma)$ is then defined by adjoining

$$\Sigma(p,f) = C(p,f) \cup \{p_a,\ a \in A \text{ s.t. } a \text{ is an algebraic singularity}\}$$

$$\downarrow \pi \qquad \pi(p_a) = a.$$

$$\Sigma$$

$ev: C(p,f) \to \mathbb{C}$ extends to

$$\Sigma(p,f) \to \mathbb{P}^1 \quad \text{(meromorphic fn.)}$$

$$ev(p_a) = \lim_{\substack{z \to a \\ z \in \mathbb{D}^\times(0,1) \underset{\psi}{\simeq} \pi^{-1}(\mathbb{D}^\times(a))}} ev(\psi(z))$$

In case Σ is compact and $f \in \mathcal{O}_p(\Sigma)$ is s.t. there are only finitely many algebraic singularities of f, $\Sigma(p,f)$ is compact and π is a branched covering map – branched at $\Sigma(p,f) \setminus C(p,f)$.

$$\begin{tikzcd} C(p,f) \arrow[d, "\text{covering map}"'] \arrow[r, phantom, "\subset"] & \Sigma(p,f) \arrow[d, "\pi"', "\text{surjective}"] \\ \Sigma \setminus A \arrow[r, phantom, "\subset"] & \Sigma \end{tikzcd}$$

$A \subset \Sigma$ finite set of alg. sing. of f

§3. Example. (1) - $\Sigma = \mathbb{P}^1$, f = germ of $\sqrt{z}$ near $z=1$

$$= (1+(z-1))^{\frac{1}{2}} = \sum_{n=0}^{\infty} (z-1)^n \cdot \binom{1/2}{n} \in \mathcal{O}_1(\mathbb{P}^1).$$

Then $\mathcal{A} = \{0, \infty\}$,

$$\begin{array}{ccccc} C(1,\sqrt{z}) = \mathbb{C}^\times & & \subset & \Sigma(1,\sqrt{z}) = \mathbb{P}^1 \\ \downarrow \pi & (\pi(w) = w^2) & & \downarrow \\ \mathbb{C}^\times = \Sigma \setminus \mathcal{A} & & \subset & \Sigma = \mathbb{P}^1 \end{array}$$

(2) Let $f(z) = (z-a_1) \cdots (z-a_n)$; $a_1, \dots, a_n \in \mathbb{C}$ are distinct.

Pick $\alpha \in \mathbb{C} \setminus \{a_1, \dots, a_n\}$ and consider a branch of $\sqrt{f(z)}$ defining a local germ at $\mathcal{O}_\alpha(\mathbb{P}^1)$.

$\mathcal{A} = \{a_1, \dots, a_n\} \cup \{\infty\}$.

$$\begin{array}{cl} C & = \text{unbranched R.S.} \\ \downarrow & \quad 2:1 \text{ cover} \\ \mathbb{P}^1 \setminus \mathcal{A} & \end{array}$$

For each $j \in \{1, \dots, n\}$, let $D_j \subset \mathbb{C}$ be a small disc around a_j s.t. $D_j \cap \mathcal{A} = \{a_j\}$. (say $D_j = D(a_j, r_j)$)

$f(z) = (z-a_j) \cdot h_j(z)$ $\qquad h_j(z) = \prod_{\substack{k \neq j \\ 1 \leq k \leq n}} (z-a_k) \neq 0$ on D_j

simply-connected

Hence $\exists$ a hol. fn. $g_j(z) : D_j \to \mathbb{C}$ s.t. $g_j(z)^2 = h_j(z)$.

So, $\forall$ $w = a_j + \rho e^{i\theta}$ $(0 < \rho < r_j)$, we have a local fn. germ $\varphi_w \in \mathcal{O}_w(\mathbb{P}^1)$ s.t. $\varphi_w^2 = f(w)$ - namely $\varphi_w = \underline{\sqrt{\rho}\, e^{i\theta/2}}\, g_j$

$$\Rightarrow \quad \pi^{-1}(D_j \setminus \{a_j\}) \cong \mathbb{D}^\times \xrightarrow[z \mapsto z^2]{} \mathbb{D}^\times \cong D_j \setminus \{a_j\}.$$

Near ∞: Let $R > \max\{|a_j| : 1 \leq j \leq n\}$ and

$$\mathcal{U}^* = \{z \in \mathbb{C} : |z| > R\} \quad \text{punctured nbd. of } \infty.$$

$$\mathcal{U} = \mathcal{U}^* \cup \{\infty\}$$

On $\mathcal{U}$, we can write $f(z) = z^n F$ where $F : \mathcal{U} \to \mathbb{C}$ does not vanish anywhere – hence has a square root.

(i) If n is odd, $f(z) = z \cdot h(z)^2$. In this case $\pi^{-1}(\mathcal{U}^*)$ is a connected 2-to-1 cover of $\mathcal{U}^*$ – hence ∞ is a branch point of index 2.

(ii) if n is even, $f(z) = h(z)^2$, hence $\pi^{-1}(\mathcal{U}^*)$ is disconnected 2-to-1 cover – i.e., ∞ is not a branch point.

$$\Sigma = \text{ramified R.S. of } \sqrt{f(z)}$$

$$\Sigma \xrightarrow{\;2:1 \text{ branched cover}\;} \mathbb{P}^1$$

$$\beta = \text{Total branching number} = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

By Riemann-Hurwitz formula,

$$2 - 2g(\Sigma) = \underset{\substack{\uparrow \\ \text{degree} \\ \text{of } \pi}}{2}\,\big(2 - 2(\underset{\substack{\uparrow \\ \text{genus} \\ \text{of } \mathbb{P}^1}}{0})\big) - \beta$$

$$\Rightarrow g(\Sigma) = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n-2}{2} & \text{if } n \text{ is even} \end{cases}$$