Lecture 31

§1. Recall - last time we defined Riemann surface associated to a local germ.

$$X : \text{Riemann surface}, \quad x_0 \in X \text{ and } \xi_0 \in \mathcal{O}_{x_0}(X) \quad \rightsquigarrow \quad \begin{array}{c} \Sigma(x_0, \xi_0) \xrightarrow{ev} \mathbb{P}^1 \text{ meromorphic.} \\ \pi \downarrow \\ X \end{array}$$

Assume X is compact. Then $\Sigma(x_0, \xi_0)$ is compact if and only if ξ_0 is algebraic (see §2 below)

§2. Lemma. - Let $c_1, c_2, \ldots, c_n : D(0,R) \to \mathbb{C}$ be holomorphic.

If $w_0 \in \mathbb{C}$ is a simple zero of $\sum_{j=0}^{n} c_j(0)\, T^{n-j} = 0$, then
$(c_0 \equiv 1)$

$\exists\ \varphi : D(0,r) \to \mathbb{C}$ hol. $(0 < r \leq R)$ s.t. $\varphi(0) = w_0$ and

$$\sum c_j\, \varphi^{n-j} \equiv 0 \text{ on } D(0,r).$$

Proof. - Let $F(z,w) = \sum_{j=0}^{n} c_j(z)\, w^{n-j}$. Let $\varepsilon > 0$ be s.t. w_0 is the only zero of $F(0,w)$ in $\overline{D(w_0, \varepsilon)}$. By continuity of F, $\exists\ 0 < r \leq R$ s.t. F does not vanish on $\{(z,w) : |z| < r,\ |w - w_0| = \varepsilon\}$

$$\forall\ z_0 \in D(0,r), \quad n(z_0) = \frac{1}{2\pi i} \int_{C(w_0,\varepsilon)} \frac{1}{F(z_0,w)} \left(\frac{\partial}{\partial w} F(z_0,w) \right) dw$$

$$= \#\text{ of zeroes of } F(z_0, w) = 0 \text{ within } D(w_0, \varepsilon)$$

We know $n(0) = 1$, and $n(z_0) \in \mathbb{Z}$ depends continuously on z_0, hence $n(z) = 1 \quad \forall\, z \in D(0,r)$.

i.e. $\forall\, z_1 \in D(0,r)$, $\exists!\, w_1 \in D(w_0,\varepsilon)$ s.t. $F(z_1,w_1)=0$

Moreover, this unique solution is given by Cauchy's integral formula

$$\varphi(z) = \frac{1}{2\pi i} \int_{C(w_0,\varepsilon)} \frac{w\, F_w(z,w)}{F(z,w)}\, dw$$

Hence $\varphi : D(0,r) \to \mathbb{C}$ is holomorphic and $F(z,\varphi(z))=0$ $\forall\, z \in D(0,r)$. $\square$

<u>Cor.</u> Let $\mathcal{O}_x$ be the ring of germs of hol. fns. at a point $x \in X$, for a Riemann surface X.

Let $P(T) = T^n + c_1 T^{n-1} + \dots + c_n \in \mathcal{O}_x[T]$. be s.t. $T^n + c_1(x) T^{n-1} + \dots + c_n(x) = 0$ has n distinct roots (say $w_1, \dots, w_n$). Then

$\exists\, \varphi_1, \dots, \varphi_n \in \mathcal{O}_x$ s.t. $\varphi_j(x) = w_j$ and

$$P(T) = \prod_{j=1}^{n} (T - \varphi_j)$$

§3. Theorem.- Let X be a Riemann surface,

$$\mathcal{M}(X) = \text{field of meromorphic fns. on } X$$
$$= \mathcal{H}ol\,(X, \mathbb{P}^1)$$

Let $P(T) = T^n + c_1 T^{n-1} + \dots + c_n \in \mathcal{M}(X)[T]$ be an irred. polynomial. If $x_0 \in X$ and $\xi_{x_0} \in \mathcal{O}_{x_0}$ solves $p_{x_0}(\xi_{x_0}(x_0)) = 0$ then the (ramified) R.S. associated to ξ_{x_0} is n-to-1 cover of X.

Hence, if X is cpct, then the Riemann surface of an algebraic germ is cpct.

<u>Proof</u>.- Let $A \subset X$ be the (discrete) set of poles of $c_1, \dots, c_n$ and $\{x \in X : p_x(T) = \sum_{j=0}^{n} c_j(x) T^{n-j}$ has multiple roots$\}$.

$\forall\; x \in X \setminus A$, $\quad p_x(T) = 0$ has n distinct roots.

Let $Y \subset |\mathcal{O}(X)|$ consist of all (p, ξ) s.t. $p \in X \setminus A$ and
$$\sum c_j(p) \cdot \xi(p)^{n-j} = 0.$$

$$\begin{array}{c} Y \\ \downarrow \\ X' \end{array} \text{ is } n\text{-to-1 covering map}$$

We claim Y is connected, hence Y = (connected component of $(x_0, \xi) \in |\mathcal{O}|$) $\downarrow$ (unramified) Riemann surface of any local soln. $p_{x_0}(\xi(x_0)) = 0$, $x_0 \in X \setminus A$.

<u>Proof that Y is connected</u> if
$$\begin{array}{c} Y = Y_1 \sqcup Y_2 \dots \sqcup Y_K \\ \downarrow \\ X' \end{array} \text{(connected components)}$$

and $\begin{matrix} Y_j \\ \pi_j \downarrow \\ X' \end{matrix}$ is n_j-to-1 cover, then ~~[illegible]~~, we can define holomorphic functions

$$a_1, \ldots, a_{n_j} : X' \to \mathbb{C} \quad \text{s.t.}$$

$$\sum a_r(x) \cdot \xi(x)^{n_j - r} = 0 \quad \forall \quad (x, \xi) \in Y_j.$$

(for $x \in X'$, let $\pi_j^{-1}(x) = (x, \xi_1), \ldots, (x, \xi_{n_j})$.

Define $e_p(x) =$ p-th elementary symmetric fn. on $\xi_1(x), \ldots, \xi_{n_j}(x)$

Check: $e_1, \ldots, e_{n_j} : X' \to \mathbb{C}$ are holomorphic functions

and using $\prod_{r=1}^{n_j} T - \xi_r(x) = \sum_{\ell=0}^{n_j} T^{n_j - \ell} (-1)^\ell e_\ell(\xi_1, \ldots \xi_{n_j})$.)

But this implies $\sum_{r=0}^{n_j} a_r T^{n_j - r}$ divides $\sum_{j=0}^{n} c_j T^{n-j}$ contradicting its irreducibility. $\square$

<u>Ex</u>: Verify that points of $a \in A \subset X$ are all algebraic singularities.

§4. <u>Puiseux Series.</u> - Let $\mathbb{C}\{\{z\}\}$ denote the field of Laurent series in z, convergent in some punctured disc

Let $F(z, w) = \sum_{j=0}^{n} a_j(z) \, w^{n-j} \in \mathbb{C}\{\{z\}\}[w]$ be irreducible.

$(a_0 = 1)$

Let $r>0$ be s.t. $a_1(z), \dots, a_n(z)$ are mero. fns. on $D(0,r)$; and $\forall\ z_0 \in D(0,r)\setminus\{0\}$, $F(z_0,w)=0$ has n-distinct roots. Then the Riemann surface associated to any local solution of $F(z,w)=0$ is n-sheeted (connected) cover of $D(0,r)$ - branched over 0:

$$\begin{array}{ccc} \Sigma & \cong & D(0,1) \ni z \\ \downarrow & & \downarrow \\ D(0,r) & \cong & D(0,1) \ni z^n \end{array}.$$

i.e. we have an iso:

$$\alpha : D(0,\rho) \xrightarrow{\sim} \begin{array}{c} \Sigma \\ \downarrow \pi \\ D(0,r) \end{array} \qquad \pi(\alpha(x)) = x^n \quad (\rho = r^{1/n}).$$

Hence $\quad F(x^n, \varphi(x)) = 0$, where $\varphi = ev \circ \alpha : D(0,\rho) \to \mathbb{C}$ mero.

<u>Theorem</u> (Puiseaux). Given $F(z,w) = w^n + a_1(z) w^{n-1} + \dots + a_n(z) \in \mathbb{C}\{\{z\}\}[w]$ irreducible, $\exists\ \varphi(\zeta) = \sum_{\ell=-k}^{\infty} c_\ell \zeta^\ell \in \mathbb{C}\{\{\zeta\}\}$ s.t.

$$F(\zeta^n, \varphi(\zeta)) = 0$$

Remarks.- (1) if $a_1, \dots, a_n$ are holomorphic on a disc around 0, then so is $\varphi(\zeta)$. In this case $ev : \Sigma \to \mathbb{C}$ considered above is 'holomorphic' at 0.

(2) Another statement of the theorem is that $F(z,w)=0$ can be solved by a Puiseux series $w = \varphi(z^{1/n})$.

(3) Let $\omega = e^{2\pi i/n}$. For each $k = 0, 1, \ldots, n-1$; replacing ζ by $\omega^k \zeta$ in $F(\zeta^n, \varphi(\zeta)) = 0$ gives $F(\zeta^n, \varphi(\omega^k \zeta)) = 0$.

Hence $\{\varphi(\omega^k \zeta) : k = 0, \ldots, n-1\}$ are n distinct roots of $F(\zeta^n, \varphi) = 0$. Thus

$$\begin{array}{ccc} \mathbb{C}\{\{z\}\} & \hookrightarrow & \mathbb{C}\{\{\zeta\}\} \\ & z = \zeta^n & \end{array}$$

is a splitting extension of $F(z, w)$.

§5. For the converse of Thm §3, let X be a Riemann surface and (x_0, ξ) a local germ whose associated R.S. $\begin{array}{c}\Sigma \\ \downarrow \pi \\ X\end{array}$ is a (branched) n-to-1 cover.

That is, we are assuming $\exists$ a discrete set of alg. singularities $A \subset X$

s.t. $\begin{array}{c}\Sigma \setminus \pi^{-1}(A) \\ \downarrow \pi \\ X \setminus A\end{array}$ is n-to-1 cover.

For $x \in X \setminus A$, let $\pi^{-1}(x) = \{(x, \xi_1), \ldots, (x, \xi_n)\}$. Set,

$\forall\ 1 \leq k \leq n$: $\quad e_k(x) = \displaystyle\sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi_{i_1}(x) \cdots \xi_{i_k}(x)$ = k-th elementary symmetric fn. in $\xi_1(x), \ldots, \xi_n(x)$

$e_1, \dots, e_n : X \setminus A \to \mathbb{C}$ are holomorphic and since A consists entirely of alg. singularities, $e_1, \dots, e_n : X \dashrightarrow \mathbb{C}$ meromorphic

(It is clear that, given $x_0 \in X \setminus A$; $e_1, \dots, e_n : \underset{x_0}{\underset{\cup}{\mathcal{U}}} \to \mathbb{C}$ are holomorphic in a nhd. of x_0. We claim that these continue analytically to single-valued fns. on $X \setminus A$ - as defined above. So, let $x_1 \in X \setminus A$ and $\gamma : [0,1] \to X \setminus A$ be a path joining x_0 and x_1. Analytic continuation along γ sets up a bijection between $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$ and hence

$$e_K (An_\gamma(\xi_j)(x_1) : 1 \leq j \leq n) = e_K (\eta_j (x_1) : 1 \leq j \leq n) \quad \text{since.}$$

$$\pi^{-1}(x_0) = \{ (x_0, \xi_1), \dots, (x_0, \xi_n) \} \qquad \pi^{-1}(x_1) = \{ (x_1, \eta_1), \dots, (x_1, \eta_n) \}$$

$$\exists \ \sigma \in S_n \ \text{ s.t. } \ An_\gamma(\xi_j) = \eta_{\sigma(j)}. \quad)$$

By construction $P(x, \xi) = 0 \quad \forall \ (x, \xi) \in \Sigma \setminus A$ where

$$P(x, T) = \sum_{j=0}^{n} T^{n-j} (-1)^j e_j(x) \qquad . \qquad \square$$