Lecture 32

§1. Differential forms.- Let M be a Riemann surface. As an oriented, smooth manifold, one has the usual notion of k-forms on M, $0 \leqslant k \leqslant 2$.

Notation. $\Omega^k(M; \mathbb{R} \text{ or } \mathbb{C}) =$ $\mathbb{R}$, or $\mathbb{C}$-valued, smooth k-forms on M.

Thus, $\Omega^0(M; \mathbb{R}) = C^\infty(M) =$ ring of smooth $\mathbb{R}$-valued fns. on M.

A 1-form locally is given by $f(x,y)\,dx + g(x,y)\,dy$ where

$(x,y): \mathcal{U} \to \mathbb{D} \subset \mathbb{R}^2$ coordinate chart

$f, g : \mathcal{U} \to \mathbb{R}$ or $\mathbb{C}$ smooth fns.

(Precise defn. – optional. $\forall\, p \in M$, let $\mathcal{O}_p =$ local ring of germs of hol./smooth fns. near p.

$$\mathfrak{m}_p = \{[f] : \underset{=0}{f(p)}\} \subset \mathcal{O}_p \quad \text{unique max'l ideal}$$

Then $\mathfrak{m}_p / \mathfrak{m}_p^2$ is 2-dim'l – spanned by $\{dx, dy\}$ for

$$(x,y) : \underset{\substack{\cup\!\!\!\shortmid \\ p}}{\mathcal{U}} \xrightarrow{\simeq} \underset{\substack{\cup\!\!\!\shortmid \\ 0}}{\mathbb{D}} \quad \text{local coordinates.}$$

$$p \longmapsto 0$$

Similar to the construction of $|\mathcal{O}| = \bigsqcup_{p \in M} \mathcal{O}_p$, we have

$$T^*M \simeq \bigsqcup_{p \in M} \mathfrak{m}_p / \mathfrak{m}_p^2$$ with suitable topology – as for $|\mathcal{O}|$ –

A 1-form on M is then a $\overbrace{\text{section}}^{\text{smooth}}$ $s: M \to T^*M$.

(i.e. $\pi \circ s = Id_M$))

§2. Complex structure on M. – Let $p \in M$ and let

$$z = x + iy : \mathcal{U} \xrightarrow{\simeq} \mathbb{D} \quad \text{be a local coordinate chart near } p.$$
$$p \mapsto 0$$

$dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$ give another basis of T_p^*M.

$$\omega = a\,dx + b\,dy \qquad ; \quad a, b : \mathcal{U} \to \mathbb{R} \text{ or } \mathbb{C} \text{ smooth}$$
$$= \frac{a - ib}{2}\,dz + \frac{a + ib}{2}\,d\bar{z}$$

Remark.- Chain rule for differentiation, and the fact that transition fns are holomorphic, imply that $\{dz, d\bar{z}\}$ is a coordinate-independent splitting of T_p^*M. Meaning:

if $z : \mathcal{U} \xrightarrow{\simeq} \mathbb{D}$ and $w : \mathcal{U} \xrightarrow{\simeq} \mathbb{D}$ are 2 local coordinates near p, related by $w = g(z) = a(x,y) + i\,b(x,y)$ where $z = x + iy$, then

$$dw = \underbrace{\left(\frac{a_x + b_y}{2} - \frac{i}{2}(a_y - b_x)\right)}_{\frac{\partial g}{\partial z}} dz + \underbrace{\left(\frac{a_x - b_y}{2} + \frac{i}{2}(a_y + b_x)\right)}_{\frac{\partial g}{\partial \bar{z}} = 0 \text{ by Cauchy-Riemann equations.}} d\bar{z}$$

Note
$$\partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)$$
$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\,\partial_y)$$

Hence $\partial_x^2 + \partial_y^2 = 4\,\partial_z\,\partial_{\bar{z}}$.
(Laplacian)

Definition .- Holomorphic differentials on M are differentials of the form $u(z)\,dz$, with u holomorphic

$z : \mathcal{U} \to \mathbb{D}$ local coordinate chart

This definition is independent of the choice of local coordinate - see remark above

$$H\Omega^1(M;\mathbb{C}) = \{ \omega \in \Omega^1(M;\mathbb{C}) \mid \forall p \in M,\ z : \mathcal{U} \cong \mathbb{D} \text{ local coord.},\ p \mapsto 0,\ \omega|_p = u(z)\,dz \text{ where } u : \mathcal{U} \to \mathbb{C} \text{ is hol.} \}$$

Similarly we can define meromorphic differentials.

$M\Omega^1(M;\mathbb{C})$. ω is a meromorphic differential if $\exists$ a discrete set $A \subset M$ s.t. $\omega \in H\Omega^1(M \setminus A;\mathbb{C})$ and locally near a point $\omega = u(z)\,dz$ for u meromorphic.

Note - if $w = g(z)$ is a hol. transition fn. for 2 local charts $z, w : \mathcal{U} \cong \mathbb{D}$, then

$$a(w)\,dw = a(g(z)) \cdot g'(z)\,dz$$

Ex. if ω is a meromorphic differential on M and $p \in M$, then $\operatorname{Res}(\omega;p)$ = coefficient of z^{-1} in the expression

$$\omega|_p = \Big(\sum_{n=-N}^{\infty} a_n z^n \Big)\,dz$$

is well-defined.

§3. Let $d : \Omega^0 \to \Omega^1 \to \Omega^2$ denote the de Rham differential. Thus for $f \in \Omega^0$, $df = f_x\, dx + f_y\, dy$ locally

$$= f_z\, dz + f_{\bar z}\, d\bar z .$$

$\omega \in \Omega^1$, $\quad \omega = f\, dx + g\, dy = u\, dz + v\, d\bar z$

$$d\omega = (g_x - f_y)\, dx \wedge dy$$

$$= (v_z - u_{\bar z})\, dz \wedge d\bar z .$$

The following is an analogue of Cauchy's theorem

Prop. - Let ω be a holomorphic differential on M. Then $d\omega = 0$

(1) If $N \subset M$ is a compact set s.t. ∂N is piecewise smooth.

then $\displaystyle\int_{\partial N} \omega = 0$.

(2) $\displaystyle\int_{\gamma} \omega$ depends only on the homology class of γ.

(3) If c is a curve in M, with initial point p and terminal point q, then

$$\int_c \omega = An_c(f_0)(q) - f_0(p) \quad \text{where}$$

$f_0(z)$ near $p = c(0)$ is local hol. fn. s.t. $\omega|_p = f_0(z)\,dz$.

$An_c(f_0)$ = analytic continuation of f_0 along c.

§4. For meromorphic differentials we have the following.

Let ω be a mero. diff'l on M with poles at $\{p_j\}_{j\in J} \subset M$.

Then $$\operatorname{Res}(\omega; p_j) = \frac{1}{2\pi i}\int_{c_j} \omega$$ c_j - small loop around p_j.

and $$\frac{1}{2\pi i}\int_{\partial N} \omega = \sum_{p\in N} \operatorname{Res}(\omega; p)$$ ($N \subset M$ cpct s.t. ∂N is piecewise smooth & $\{p_j\}_{j\in J} \cap \partial N = \phi$.)

In particular, if M is compact, then

$$\sum_{p\in M} \operatorname{Res}(\omega; p) = 0 .$$

Argument principle.- $$\frac{1}{2\pi i}\int_{\partial N} \frac{df}{f} = \#\{p\in N : f(p)=0\} - \#\{p\in N : f \text{ has a pole at } p\}$$ (counted with multiplicity).