Frobenius and the group determinant

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Main references

- Pavel Etingof et al. *Introduction to representation theory*, Student Mathematics Library volume 59, AMS publications (2010).
- Leonard Eugene Dickson *An elementary exposition of Frobenius' theory of group characters and group determinants*, Annals of Mathematics, second series, vol. 4, no. 1 (1902).
- Thomas Hawkins *The origins of the theory of group characters*, Archive for history of exact sciences, vol. 7, no. 2 (1971).
- mathshistory.st-andrews.ac.uk

Plan of the talk

- Georg Frobenius.
- Group determinant.
- Linear factors of the group determinant (Frobenius' Theorem 1).
- Irreducible factors of the group determinant (Frobenius' Theorem 2).
- Irreducible factors vs irreducible representations.
- Example of the dihedral group.

Georg Frobenius (1849-10-26 to 1917-08-03, Berlin)

- Joined University of Berlin in 1867. Studied under Kronecker, Kummer and Weierstraß.
- Obtained his doctorate in 1870 under the supervision of Weierstraß.
- Taught in Joachimsthal Gymnasium (his high school) 1870-1874.
- 1875-1892: Eidengnössische Polytechnikum, Zürich.
- Kronecker passed away in 1891. Frobenius got appointed Kronecker chair of mathematics in University of Berlin, 1892 (strong support from Fuchs and Weierstraß).

Group determinant

Dedekind [1] (around 1886) encountered what he called "group determinant" during his investigations into finite Galois extensions.

Definition (Group Determinant)

Let G be a finite group. Consider $|G|$ many variables $\{x_g : g \in G\}$. Let $M_G(\underline{x})$ be $|G| \times |G|$–matrix (rows and columns indexed by elements of G) whose (g, h)–th entry is $x_{g^{-1}h}$.

$$\Delta_G(\underline{x}) := \mathrm{Det}(M_G(\underline{x})) \text{ polynomial in variables } x_g (g \in G).$$

Example. $G = \mathbb{Z}/2\mathbb{Z}$. Variables: x_0, x_1. $M_G(x_0, x_1) = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}$.

Hence, $\Delta_G(x_0, x_1) = x_0^2 - x_1^2 = (x_0 + x_1)(x_0 - x_1)$.

[1] Richard Dedekind 1831-10-06 to 1916-02-12, Braunschweig (Germany)

$\Delta_G = \mathrm{Det}((x_{g^{-1}h})_{g,h\in G})$ Group determinant

Example. $G = \mathbb{Z}/3\mathbb{Z}$. Variables: x_0, x_1, x_2.

$$M_G(x_0, x_1, x_2) = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_1 \\ x_1 & x_2 & x_0 \end{bmatrix}$$

$\Delta_G(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2$.

$$\Delta_G(x_0, x_1, x_2) = (x_0 + x_1 + x_2)(x_0 + \omega_3 x_1 + \omega_3^2 x_2)(x_0 + \omega_3^2 x_1 + \omega_3 x_2)$$

where $\omega_3 = \exp\left(\frac{2\pi\iota}{3}\right)$.

$\Delta_G = \mathrm{Det}((x_{g^{-1}h})_{g,h\in G})$ Group determinant

Example. $G = \mathbb{Z}/N\mathbb{Z}$ $(N \geq 2)$. Variables: $x_0, x_1, \ldots, x_{N-1}$.

$$M_{\mathbb{Z}/N\mathbb{Z}}(x_0, \ldots, x_{N-1}) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{N-2} & x_{N-1} \\ x_{N-1} & x_0 & \cdots & x_{N-3} & x_{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & \cdots & x_{N-1} & x_0 \end{bmatrix}$$

$$\Delta_{\mathbb{Z}/N\mathbb{Z}} = \prod_{k=0}^{N-1} \left(x_0 + \omega_N^k x_1 + \omega_N^{2k} x_2 + \cdots + \omega_N^{(N-1)k} x_{N-1} \right)$$

where $\omega_N = \exp\left(\frac{2\pi\iota}{N}\right)$.

Group determinant for abelian groups

Theorem (Dedekind)

If G is a finite abelian group, then

$$\boxed{\Delta_G(\underline{x}) = \prod_{\substack{\chi : G \to \mathbb{C}^\times \\ \text{group homomorphism}}} \left(\sum_{g \in G} \chi(g) x_g \right)}$$

- When $G = \mathbb{Z}/N\mathbb{Z}$ written as $\langle \sigma | \sigma^N = e \rangle$, there are exactly N group homomorphisms $\chi_k : G \to \mathbb{C}^\times$ $(0 \leq k \leq N-1)$, given by: $\chi_k(\sigma) = \omega_N^k$.
- Using the structure theorem of finite abelian groups (Kronecker (1870)), it follows that for any finite abelian group G: $\left|\mathrm{Hom}_{\mathrm{gp}}(G, \mathbb{C}^\times)\right| = |G|$.

Dedekind-Frobenius correspondences, 1896

Dedekind wrote to Frobenius (March 25, 1896) a letter containing the definition of the group determinant and the factorization in the abelian case. Dedekind also hinted at some computations in the non-abelian case that he had done (without including them). Upon Frobenius' insistence, he hesitatingly formulated a conjecture in a letter dated April 3, 1896, for an arbitrary finite group G.

Conjecture (Dedekind)

Number of distinct linear factors in $\Delta_G(\underline{x})$ is equal to the index of the commutator subgroup $[G, G]$ (i.e, $|G|/|[G, G]|$).

(recall the commutator subgroup $[G, G]$ is the (normal) subgroup generated by $aba^{-1}b^{-1}$ for all $a, b \in G$).
Dedekind ended the letter inviting Frobenius to pursue this conjecture:

I would be delighted if you wished to involve yourself with these matters, because I distinctly feel that I will not achieve anything here.

Dedekind-Frobenius correspondences, 1896

- Within 10 days, Frobenius managed to prove this conjecture. He presented his research to Berlin academy on July 30, 1896 titled "Über Gruppencharaktere".
- By the end of the year (December 3, 1896) Frobenius had obtained very deep results about factorization of $\Delta_G(\underline{x})$. He published these in "Über die Primfactoren der Gruppendeterminante".

Convention. Note that $\Delta_G(\underline{x})$ is homogeneous of degree $N = |G|$. Also, if x_e is the variable corresponding to the neutral element $e \in G$, then the coefficient of x_e^N in $\Delta_G(\underline{x})$ is 1. This is simply because the diagonal entries of $M_G(\underline{x})$ are all equal to x_e.
Here, and for the rest of this talk, a factor $p(\underline{x})$ of $\Delta_G(\underline{x})$ (necessarily homogeneous) will always assumed to be **monic with respect to the variable** x_e (that is, the coefficient of $x_e^{\deg(p)}$ is 1).

Frobenius' Theorem 1 (July, 1896)

Theorem (Frobenius)

Linear factors in $\Delta_G(\underline{x})$ are

$$\left\{ \ell_\chi(\underline{x}) = \sum_{g\in G} \chi(g)x_g \text{ where } \chi : G \to \mathbb{C}^\times \text{ is a group homomorphism} \right\}.$$

Moreover, each such factor appears with multiplicity 1.

Remark

Note that, if $\chi : G \to \mathbb{C}^\times$ is a group homomorphism, then for every $a, b \in G$ we have: $\chi(aba^{-1}b^{-1}) = \chi(a)\chi(b)\chi(a)^{-1}\chi(b)^{-1} = 1$. Hence, $\chi([G, G]) = \{1\}$.
Furthermore, $G/[G, G]$ is abelian. All of this implies that number of linear factors in $\Delta_G(\underline{x})$ is equal to

$$|\mathrm{Hom}_{gp}(G, \mathbb{C}^\times)| = |\mathrm{Hom}_{gp}(G/[G, G], \mathbb{C}^\times)| = |G/[G, G]|$$

Proof of Frobenius' Theorem 1

Let $\chi : G \to \mathbb{C}^\times$ be a group homomorphism. $\ell_\chi(\underline{x}) := \sum_{g \in G} \chi(g) x_g$.

To prove: $\ell_\chi(\underline{x})$ divides $\Delta_G(\underline{x})$ with multiplicitly 1.

For $h \in G$, let Column(h) denote the h-th column of $M_G(\underline{x})$. Replace Column(e) by $\sum_{h \in G} \chi(h)$Column(h).

$$M_G(\underline{x}) \rightsquigarrow \begin{bmatrix} \ell_\chi(\underline{x}) & * & \cdots & * \\ \vdots & * & \cdots & * \\ \chi(g)\ell_\chi(\underline{x}) & * & x_{g^{-1}h} & * \\ \vdots & * & \cdots & * \end{bmatrix}$$

Because, g-th entry of Column(e) becomes:

$$\sum_{h \in G} \chi(h) x_{g^{-1}h} = \sum_{\sigma \in G} \chi(g\sigma) x_\sigma = \sum_{\sigma \in G} \chi(g)\chi(\sigma) x_\sigma = \chi(g)\ell_\chi(\underline{x}).$$

$\ell_\chi(\underline{x}) = \sum_{g\in G} \chi(g)x_g$ divides $\Delta_G(\underline{x})$ only once

Hence, $\Delta_G(\underline{x}) = \ell_\chi(\underline{x}) \cdot \mathrm{Det}(A)$,

where $A = \begin{bmatrix} 1 & * & \cdots & * \\ \vdots & * & \cdots & * \\ \chi(g) & * & x_{g^{-1}h} & * \\ \vdots & * & \cdots & * \end{bmatrix}$.

Row operation on A: Replace $\mathrm{Row}(g)$ by $\mathrm{Row}(g) - \chi(g)\mathrm{Row}(e)$, for every $g \neq e$.

$$A \rightsquigarrow \begin{bmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & * & \boxed{a_{g,h}} & * \\ 0 & * & \cdots & * \end{bmatrix}, \quad \begin{aligned} a_{g,h} &= x_{g^{-1}h} - \chi(g)x_h \\ &= \chi(gh^{-1})(\chi(g^{-1}h)x_{g^{-1}h} - \chi(h)x_h) \end{aligned}$$

Hence, $\frac{\Delta_G(\underline{x})}{\ell_\chi(\underline{x})} = \mathrm{Det}(A)$ depends only on $\chi(a)x_a - \chi(b)x_b$.

$\ell_\chi(\underline{x}) = \sum_{g \in G} \chi(g) x_g$ divides $\Delta_G(\underline{x})$ only once

Fact. Let $P(w_a - w_b : 1 \leq a, b \leq n)$ be a (non-zero) polynomial in n variables, depending only on the differences of variables, as indicated. Then P is not divisible by $\sum_a w_a$.

(For a proof of this fact, replace w_a by $w_a + \frac{t}{n}$. This does not change P, but adds t to $\sum_a w_a$. Assuming the contrary, we arrive at a linear polynomial in t dividing something independent of t, which is absurd.)

The proof of this part is finished by taking $w_a = \chi(a) x_a$ $(a \in G)$ and $P = \frac{\Delta_G(\underline{x})}{\ell_\chi(\underline{x})} = \mathrm{Det}(A)$.

Every linear factor of $\Delta_G(\underline{x})$ is of the form $\ell_\chi(\underline{x})$

Brilliant idea. Consider three sets of variables $\underline{x} = \{x_g : g \in G\}$, $\underline{y} = \{y_g : g \in G\}$ and $\underline{z} = \{z_g : g \in G\}$ related by:

$$\underline{z} = \underline{x} * \underline{y} \text{ meaning } z_g = \sum_{\substack{a,b \in G \\ ab=g}} x_a y_b = \sum_{a \in G} x_a y_{a^{-1}g}.$$

Then, $\boxed{M_G(\underline{z}) = M_G(\underline{x}) \cdot M_G(\underline{y})}$.

Proof. For $g, h \in G$, the (g,h)–th entry of $M_G(\underline{z})$ is given by:

$$z_{g^{-1}h} = \sum_{a \in G} x_a y_{a^{-1}g^{-1}h} = \sum_{c \in G} x_{g^{-1}c} y_{c^{-1}h} = \big(M_G(\underline{x}) M_G(\underline{y})\big)_{g,h}.$$

Every linear factor of $\Delta_G(\underline{x})$ is of the form $\ell_\chi(\underline{x})$

$$\boxed{\underline{z} = \underline{x} * \underline{y} \equiv \{z_g = \sum_a x_a y_{a^{-1}g}\}_{g\in G} \;\Rightarrow\; M_G(\underline{z}) = M_G(\underline{x}) M_G(\underline{y})}$$

Taking determinant, we get $\Delta_G(\underline{z}) = \Delta_G(\underline{x})\Delta_G(\underline{y})$.

Now, assume that there is a linear form $\ell(\underline{x}) = \sum_g \lambda_g x_g$, with $\lambda_e = 1$, which divides $\Delta_G(\underline{x})$.

Claim. $\ell(\underline{z}) = \ell(\underline{x})\ell(\underline{y})$.

Note. Comparing coefficients of $x_a y_b$ on both sides, we get $\lambda_{ab} = \lambda_a \lambda_b$. That is, $g \mapsto \lambda_g$ is a group homomorphism, and $\ell = \ell_\lambda$ as desired.

Every linear factor of $\Delta_G(\underline{x})$ is of the form $\ell_\chi(\underline{x})$

$$\left\{ z_g = \sum_{a\in G} x_a y_{a^{-1}g} \right\}_{g\in G}, \qquad \Delta_G(\underline{z}) = \Delta_G(\underline{x})\Delta_G(\underline{y})$$

$\ell(\underline{z}) = \sum_{g\in G} \lambda_g z_g$ divides $\Delta_G(\underline{z})$. (recall $\lambda_e = 1$).

To prove: $\boxed{\ell(\underline{z}) = \ell(\underline{x})\ell(\underline{y})}$.

Since $\ell(\underline{z})$ divides $\Delta_G(\underline{x})\Delta_G(\underline{y})$, it must be product of a linear form in $\underline{x}$ and another one in $\underline{y}$: $\ell(\underline{z}) = \ell_1(\underline{x})\ell_2(\underline{y})$.

Specializing $y_g = \delta_{g,e}$ turns $z_g = x_g$ and $\ell_2(\underline{y})$ into a complex number, say c_2. Similarly for the same specialization of $\underline{x}$ variables. We get:
$\ell(\underline{x}) = \ell_1(\underline{x})c_2, \qquad \ell(\underline{y}) = c_1\ell_2(\underline{y})$.

Put together, $\ell(\underline{x})\ell(\underline{y}) = c_1c_2\ell(\underline{z})$. But c_1c_2 is the coefficient of z_e in $\ell(\underline{z})$ assumed to be 1.

Frobenius' Theorem 2 (December 1896)

Theorem

Consider the factorization of $\Delta_G(\underline{x})$ into irreducible factors:

$$\Delta_G(\underline{x}) = \prod_{i=1}^{r} P_i(\underline{x})^{d_i}.$$

Then,

1. *r is equal to the number of conjugacy classes of G.*
2. $\deg(P_i) = d_i$. *In particular,* $\boxed{\sum_{i=1}^{r} d_i^2 = |G|}$.

Recall that conjugacy classes in G are equivalence classes under the equivalence relation: $a \sim b$ iff there exists g such that $a = gbg^{-1}$.

Example: $G = S_3$ symmetric group on 3 letters

Variables: $x_0, \ldots, x_5$ corresponding to the following ordering of permutations:

$$e, (123), (132), (23), (13), (12).$$

$$M_G(\underline{x}) = \left[\begin{array}{ccc|ccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_0 & x_1 & x_4 & x_5 & x_3 \\ x_1 & x_2 & x_0 & x_5 & x_3 & x_4 \\ \hline x_3 & x_4 & x_5 & x_0 & x_1 & x_2 \\ x_4 & x_5 & x_3 & x_2 & x_0 & x_1 \\ x_5 & x_3 & x_4 & x_1 & x_2 & x_0 \end{array}\right]$$

Dedekind computed $\Delta_G = (u+v)(u-v)(u_1 u_2 - v_1 v_2)^2$, where ($\omega = \omega_3$ here):

$$\begin{aligned} u &= x_0 + x_1 + x_2, & v &= x_3 + x_4 + x_5, \\ u_1 &= x_0 + \omega x_1 + \omega^2 x_2, & v_1 &= x_3 + \omega x_4 + \omega^2 x_5, \\ u_2 &= x_0 + \omega^2 x_1 + \omega x_2, & v_2 &= x_3 + \omega^2 x_4 + \omega x_5. \end{aligned}$$

Example: $G = S_3$

View $M_G(\underline{x})$ as a linear operator on $\mathbb{C}^6$ with (ordered) basis $\{b_0, \ldots, b_5\}$.

$$\begin{aligned} \alpha_0 &= b_0 + b_1 + b_2, & \beta_0 &= b_3 + b_4 + b_5 \\ \alpha_1 &= b_0 + \omega b_1 + \omega^2 b_2, & \beta_1 &= b_3 + \omega b_4 + \omega^2 b_5 \\ \alpha_2 &= b_0 + \omega^2 b_1 + \omega b_2, & \beta_2 &= b_3 + \omega^2 b_4 + \omega b_5 \end{aligned}$$

$$M_G(\underline{x}) \quad \underset{\alpha_0, \beta_0, \alpha_1, \beta_2, \alpha_2, \beta_1}{\overset{\text{Ordered basis}}{\rightsquigarrow}} \quad \begin{bmatrix} \boxed{\begin{matrix} u & v \\ v & u \end{matrix}} & 0 & 0 \\ 0 & \boxed{\begin{matrix} u_1 & v_2 \\ v_1 & u_2 \end{matrix}} & 0 \\ 0 & 0 & \boxed{\begin{matrix} u_2 & v_1 \\ v_2 & u_1 \end{matrix}} \end{bmatrix}$$

Example: $G = S_3$

View $M_G(\underline{x})$ as a linear operator on $\mathbb{C}^6$ with (ordered) basis $\{b_0, \ldots, b_5\}$.

$$\begin{aligned} \alpha_0 &= b_0 + b_1 + b_2, & \beta_0 &= b_3 + b_4 + b_5 \\ \alpha_1 &= b_0 + \omega b_1 + \omega^2 b_2, & \beta_1 &= b_3 + \omega b_4 + \omega^2 b_5 \\ \alpha_2 &= b_0 + \omega^2 b_1 + \omega b_2, & \beta_2 &= b_3 + \omega^2 b_4 + \omega b_5 \end{aligned}$$

$$M_G(\underline{x}) \quad \underset{\alpha_0+\beta_0,\, \alpha_0-\beta_0,\, \alpha_1,\, \beta_2,\, \beta_1,\, \alpha_2}{\overset{\text{Ordered basis}}{\rightsquigarrow}} \quad \begin{bmatrix} \boxed{u+v} & 0 & 0 & 0 \\ 0 & \boxed{u-v} & 0 & 0 \\ 0 & 0 & \boxed{\begin{matrix} u_1 & v_2 \\ v_1 & u_2 \end{matrix}} & 0 \\ 0 & 0 & 0 & \boxed{\begin{matrix} u_1 & v_2 \\ v_1 & u_2 \end{matrix}} \end{bmatrix}$$

Representation theory (1896-12-03, Berlin)

Definitions. Let G be a group. In this talk, vector spaces are over $\mathbb{C}$.

- A G–representation (V, ρ) is a vector space V together with linear maps $\rho(g) : V \to V$, for every $g \in G$, such that:

$$\rho(e) = \mathrm{Id}_V, \qquad \rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2).$$

- A subrepresentation of a representation (V, ρ) is a vector subspace $V_1 \subset V$ such that $\rho(g)(V_1) \subset V_1$, for every $g \in G$. A representation (V, ρ) is said to be irreducible if its only subrepresentations are $\{0\}$ and V.
- A G–linear map (or a G–intertwiner) between two representations (V, ρ) and (V', ρ') is a linear map $X : V \to V'$ such that

$$\rho'(g) \circ X = X \circ \rho(g), \text{ for every } g \in G.$$

(Easy check: kernel and image of a G–intertwiner are subrepresentations of V and V' respectively.)

Representation theory (1896-12-03, Berlin)

Direct Sum. Given two representations (V_1, ρ_1) of (V_2, ρ_2), their direct sum is the representation (V, ρ), where $V = V_1 \oplus V_2$ and $\rho(g) = \rho_1(g) \oplus \rho_2(g)$, for every $g \in G$. That is, $\rho(g)$ is a block diagonal matrix:

$$\rho(g) = \left[\begin{array}{cc} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{array} \right]$$

Notation. For two vector spaces V, W, $\mathrm{Hom}_{\mathbb{C}}(V, W)$ denotes the vector space of all linear maps $V \to W$. If (V, ρ) and (W, ρ') are G–representations, then $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}_{\mathbb{C}}(V, W)$ denotes the vector space of all G–intertwiners.

$$\mathrm{Hom}_G(V, W) = \{X \in \mathrm{Hom}_{\mathbb{C}}(V, W) : \rho'(g)X = X\rho(g),\ \forall\ g \in G\}$$

Examples

Remark. (V,ρ) *is a G–representation* is same as saying $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. If $n = \dim(V)$, it is same as (after picking a basis of V) a group homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$.

1-dimensional representations of G are same as group homomorphisms $G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$.

When G is finite, abelian. Every finite-dimensional, irreducible representation of G is 1-dimensional.

Proof. Let (V,ρ) be a finite-dimensional representation. For every $g \in G$, there exists $m \in \mathbb{Z}_{\geq 1}$ such that $g^m = e$. So $\rho(g)^m = \mathrm{Id}_V$, hence $\rho(g)$ is diagonalizable.

This implies that $\{\rho(g)\}_{g\in G}$ is a collection of pairwise commuting, diagonalizable matrices. Thus they can be diagonalized simultaneously, giving a joint eigenvector $0 \neq v \in V$. $\mathbb{C}v \subset V$ is a non-zero subrepresentation which will have to be equal to V, if V is irreducible.

Example of the dihedral group D_n

- D_n is the dihedral group (symmetries of a regular n–gon). It has the following presentation:

$$D_n = \left\langle s, r | s^2 = r^n = (sr)^2 = e \right\rangle, \qquad (srs = r^{-1})$$

- $|D_n| = 2n$. Its elements can be listed as (note $r^k s = sr^{-k} = sr^{n-k}$):

$$\{e, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}$$

- Let $\zeta \in \mathbb{C}$ be such that $\zeta^n = 1$. We have a 2–dimensional representation of D_n, denoted here by (V_ζ, ρ_ζ):

$$\rho_\zeta(r) = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, \qquad \rho_\zeta(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note: $V_\zeta \cong V_{\zeta^{-1}}$.
If $\zeta \neq \zeta^{-1}$, this representation is irreducible.
If $\zeta = \pm 1$ (ζ could be -1 iff n is even), V_ζ further breaks into two 1–dimensional representations.

Example: regular representation

Let G be a finite group. Let $\mathbb{C}G$ be a $|G|$-dimensional vector space, with basis $\{|g\rangle : g \in G\}$.
For each $\sigma \in G$, let $L(\sigma) : \mathbb{C}G \to \mathbb{C}G$ be defined by: $L(\sigma)\,|g\rangle = |\sigma g\rangle$.
Then $(\mathbb{C}G, L)$ is a G–representation.

Lemma

For any G–representation (V, ρ), we have: $\mathrm{Hom}_G(\mathbb{C}G, V) \cong V$.

PROOF. Any G–intertwiner $X : \mathbb{C}G \to V$ is completely determined by $v = X\,|e\rangle$. $(X\,|g\rangle = X(L(g)\,|e\rangle) = \rho(g)(X\,|e\rangle) = \rho(g)(v))$
Conversely, given $v \in V$, the map $|g\rangle \mapsto \rho(g)(v)$ is a G–intertwiner.
These assignments are inverse to each other and we are done.

Two fundamental results

Let G be a finite group. Let $\{(V_\lambda, \rho_\lambda) : \lambda \in \Lambda_G\}$ be the set of isomorphism classes of irreducible, finite-dimensional G–representations.

- Schur's lemma [2] $\dim(\mathrm{Hom}_G(V_\lambda, V_\mu)) = \delta_{\lambda\mu}$.
- Maschke's theorem [3] Let V be a finite-dimensional representation of G. Then there exist non-negative integers $\{m_\lambda(V) : \lambda \in \Lambda_G\}$ such that:

$$V \cong \bigoplus_{\lambda \in \Lambda_G} V_\lambda^{\oplus m_\lambda(V)}$$

The non-negative integers $m_\lambda(V)$ can be computed as

$$m_\lambda(V) = \dim\left(\mathrm{Hom}_G(V, V_\lambda)\right)$$

[2] Issai Schur. 1875-01-10, Magilev, Russian empire (now Belarus) to 1941-01-10, Tel Aviv, Palestine (now Israel)

[3] Heinrich Maschke. 1853-10-24, Breslau, Prussia (now Poland) to 1908-03-01, Chicago, USA

The case of the regular representation

Taking $V = \mathbb{C}G$, we get $\mathbb{C}G \cong \bigoplus_{\lambda \in \Lambda_G} V_\lambda^{\oplus d_\lambda}$,

where:

$$d_\lambda = \dim(\mathrm{Hom}_G(\mathbb{C}G, V_\lambda)) = \dim(V_\lambda).$$

Hence $|G| = \sum_{\lambda \in \Lambda_G} d_\lambda^2$. In particular Λ_G is a finite set.

Analogy with Frobenius' Theorem 2

$\Delta_G(\underline{x}) = \prod_{i=1}^r P_i(\underline{x})^{d_i}$	$\mathbb{C}G \cong \bigoplus_{\lambda \in \Lambda_G} V_\lambda^{\oplus d_\lambda}$
$d_i = \deg(P_i)$	$d_\lambda = \dim(V_\lambda)$
$\sum_{i=1}^r d_i^2 = \vert G\vert$	$\sum_{\lambda \in \Lambda_G} d_\lambda^2 = \vert G\vert$
$r = \vert\text{Conj. classes}\vert$	$\vert\Lambda_G\vert = \vert\text{Conj. classes}\vert$

From irreducible representations to factorization of $\Delta_G(\underline{x})$

- Let $(V_\lambda, \rho_\lambda)$ be a finite-dimensional, irreducible representation of G. Choose a basis $\{v_i^\lambda : 1 \leq i \leq d_\lambda\}$. $d_\lambda = \dim(V_\lambda)$.
- For each $g \in G$, $1 \leq i, j \leq d_\lambda$, let $\rho_\lambda(g)_{ij} \in \mathbb{C}$ be the matrix coefficient of $\rho_\lambda(g)$ in the basis chosen above.
- Define $\ell_{ij}^\lambda(\underline{x}) = \sum_{g \in G} \rho_\lambda(g)_{ij} x_g$.
- Let $P_\lambda(\underline{x}) = \mathrm{Det}(\ell_{ij}^\lambda(\underline{x}))$.

$$\boxed{\Delta_G(\underline{x}) = \prod_{\lambda \in \Lambda_G} P_\lambda(\underline{x})^{d_\lambda}}$$

Example of D_n (n is odd)

Variables. $x_k \leftrightarrow r^k$ and $y_k \leftrightarrow sr^k$. Here $0 \le k \le n-1$.

List of irreducible representations.

- Two 1-dimensional representations: $V_{+,\pm}$. r acts as 1 and s acts as ± 1. Linear factors coming from these:

$$\ell^{+,+} = \sum_{k=0}^{n-1} x_k + y_k, \qquad \ell^{+,-} = \sum_{k=0}^{n-1} x_k - y_k.$$

- 2-dimensional representations: (V_ζ, ρ_ζ) where $\zeta = \omega_n^j$, $1 \le j \le (n-1)/2$.

$$\rho_\zeta(r^k) = \begin{bmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{bmatrix}, \qquad \rho_\zeta(sr^k) = \begin{bmatrix} 0 & \zeta^{-k} \\ \zeta^k & 0 \end{bmatrix}.$$

$$2n = 1 + 1 + 4\left(\frac{n-1}{2}\right) \Rightarrow \text{ These are all!}$$

Example of D_n (n is odd)

$$\rho_\zeta(r^k) = \begin{bmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{bmatrix}, \qquad \rho_\zeta(sr^k) = \begin{bmatrix} 0 & \zeta^{-k} \\ \zeta^k & 0 \end{bmatrix}.$$

Linear forms coming from these (recall: $\zeta = \omega_n^j$, $1 \le j \le (n-1)/2$):

$$\ell_{11}^{(j)} = \sum_{k=0}^{n-1} \zeta^k x_k, \qquad \ell_{12}^{(j)} = \sum_{k=0}^{n-1} \zeta^{-k} y_k$$

$$\ell_{21}^{(j)} = \sum_{k=0}^{n-1} \zeta^k y_k, \qquad \ell_{22}^{(j)} = \sum_{k=0}^{n-1} \zeta^{-k} x_k$$

$$\boxed{\Delta_{D_n}(\underline{x}) = \ell^{+,+}\ell^{+,-} \prod_{j=1}^{\frac{n-1}{2}} \left(\ell_{11}^{(j)}\ell_{22}^{(j)} - \ell_{12}^{(j)}\ell_{21}^{(j)}\right)^2}$$

Example of D_n (n is even). $x_k \leftrightarrow r^k$, $y_k \leftrightarrow sr^k$

List of irreducible representations.

- Four 1-dimensional representations: $V_{\pm,\pm}$. On $V_{\epsilon,\eta}$, r acts by $\epsilon 1$ and s acts by $\eta 1$.
  Linear forms:

$$\ell^{+,+} = \sum_{k=0}^{n-1} x_k + y_k, \qquad \ell^{+,-} = \sum_{k=0}^{n-1} x_k - y_k$$

$$\ell^{-,+} = \sum_{k=0}^{n-1} (-1)^k (x_k + y_k), \qquad \ell^{-,-} = \sum_{k=0}^{n-1} (-1)^k (x_k - y_k)$$

- 2-dimensional representations (V_ζ, ρ_ζ), where $\zeta = \omega_n^j$, $1 \le j \le (n-2)/2$ (as before).

$$2n = 1 + 1 + 1 + 1 + 4\left(\frac{n-2}{2}\right) \Rightarrow \text{ These are all!}$$

Example of D_n (n is even)

$$\Delta_{D_n}(\underline{x}) = \ell^{+,+}\ell^{+,-}\ell^{-,+}\ell^{-,-}\prod_{j=1}^{\frac{n-2}{2}}\left(\ell_{11}^{(j)}\ell_{22}^{(j)} - \ell_{12}^{(j)}\ell_{21}^{(j)}\right)^2$$

where, for each $1 \leq j \leq (n-2)/2$, let $\zeta = \omega_n^j$ and define:

$$\ell_{11}^{(j)} = \sum_{k=0}^{n-1}\zeta^k x_k, \qquad \ell_{12}^{(j)} = \sum_{k=0}^{n-1}\zeta^{-k} y_k$$

$$\ell_{21}^{(j)} = \sum_{k=0}^{n-1}\zeta^k y_k, \qquad \ell_{22}^{(j)} = \sum_{k=0}^{n-1}\zeta^{-k} x_k$$

Danke Schön!