Lecture 0

Overview. – Representation theory studies (linear) actions of various "algebraic structures" (such as groups, Lie algebras, associative algebras ...) on vector spaces. The relevant "algebraic structures" often arise from geometry, combinatorics or physics; and techniques of representation theory help us understand their "irreducible or indecomposable" actions (see §1–3 below for definitions).

§1. Definitions. – Let k be a field. A unital, associative algebra, $\mathcal{A}$ over k, is a k-vector space together with a bilinear map called multiplication $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$; and a distinguished element called the unit $1_{\mathcal{A}} \in \mathcal{A}$ such that the following axioms hold:

- multiplication is associative : $\mu(\mu(a,b),c) = \mu(a,\mu(b,c))$ $\forall\, a,b,c \in \mathcal{A}$.
- 1_A is neutral : $\mu(1_A, x) = \mu(x, 1_A) = x \quad \forall\, x \in \mathcal{A}$.

Remark. – It has become customary to drop the notation μ and simply write $ab = \mu(a,b)$.

A representation of $\mathcal{A}$ is the data of a k-vector space V and a (unital) ring homomorphism $\rho : \mathcal{A} \to \mathrm{End}_k(V)$

Here, $\mathrm{End}_k(V)$ is the algebra of all k-linear maps $V \to V$.

Remark. - Again, for convenience, we often drop ρ from the notation and write $a \cdot v = \rho(a)(v)$ $\forall\, a \in \mathcal{A},\ v \in V$.

§2. A-linear maps (or A-intertwiners). Let $\mathcal{A}$ be a unital, assoc. algebra over k, and let

$$\rho_1 : \mathcal{A} \to \mathrm{End}_k(V_1)$$
$$\rho_2 : \mathcal{A} \to \mathrm{End}_k(V_2)$$

be two repns. of $\mathcal{A}$. An $\mathcal{A}$-linear map from V_1 to V_2 (also sometimes called an $\mathcal{A}$-intertwiner) is a k-linear map $f : V_1 \to V_2$ s.t.

$$f(\rho_1(a)(v_1)) = \rho_2(a)(f(v_1)) \quad \forall\, a \in \mathcal{A},\ v_1 \in V_1.$$

$$\mathrm{Hom}_{\mathcal{A}}(V_1, V_2) = \Big\{ \underset{k\text{-linear}}{f : V_1 \to V_2} \ : \ \underset{\forall\, a \in \mathcal{A},\ v_1 \in V_1}{f(a \cdot v_1) = a \cdot f(v_1)} \Big\} \subset \mathrm{Hom}_k(V_1, V_2).$$

Lemma. Let $f : V_1 \to V_2$ be an $\mathcal{A}$-linear map. Then $\mathrm{Ker}(f) \subset V_1$ and $\mathrm{Image}(f) \subset V_2$ are $\mathcal{A}$-subrepresentations*.

Proof. $\mathrm{Ker}(f) \subset V_1$: Let $v \in \mathrm{Ker}(f)$ and $a \in \mathcal{A}$. Then

$$f(a \cdot v) = a \cdot f(v) = a \cdot 0 = 0 \quad \Rightarrow \quad a \cdot v \in \mathrm{Ker}(f).$$

Hence $\mathrm{Ker}(f) \subset V_1$ is a subrepn.

$\mathrm{Im}(f) \subset V_2$: If $f(v) \in \mathrm{Im}(f)$ and $a \in \mathcal{A}$, then

$a \cdot f(v) = f(a \cdot v) \in \mathrm{Im}(f)$. Hence, $\mathrm{Im}(f)$ is a subrepn. $\square$

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* An $\mathcal{A}$-subrepresentation of V (here V is a repn. of $\mathcal{A}$) is a subspace $U \subset V$ s.t.

$$a \cdot u \in U \quad \forall\, a \in \mathcal{A},\ u \in U.$$

§3. Irreducible and indecomposable representations. Again, let $\mathcal{A}$ be a unital, associative algebra over a field k. Given two repns. V_1 and V_2 of $\mathcal{A}$, we have a natural linear action of $\mathcal{A}$ on $V_1 \oplus V_2$ - called direct sum of representations :

$$a \cdot (v_1, v_2) = (a \cdot v_1 , a \cdot v_2) \quad \forall a \in \mathcal{A},\ v_1 \in V_1,\ v_2 \in V_2.$$

An $\mathcal{A}$-representation V is said to be irreducible if the only sub-$\mathcal{A}$-repns. of V are $\{0\}$ and V.

We say V is indecomposable if $V \simeq V_1 \oplus V_2$ implies either $V_1 = \{0\}$ or $V_2 = \{0\}$. That is, V cannot be isomorphic to a direct sum of non-zero subrepresentations.

§4. Questions / Problems of representation theory :

Given a unital, associative algebra $\mathcal{A}$ over k :

(i) Complete reducibility question : is every representation isomorphic to a direct sum of irreducible representations ?
(the answer often depends on finite-dimensionality of repns., and char(k), even sometimes whether k is algebraically closed, or not)

(ii) Classify irreducible / indecomposable repns. of $\mathcal{A}$. (classification problem)

(iii) Compute dimensions (or more generally <u>characters</u>) of irreducible $\mathcal{A}$-repns. - More ambitiously - give <u>explicit construction</u> of these representations.

§5. <u>Some examples.</u> - (1) <u>$\mathcal{A} = k$.</u> A representation, in this case, is just a k-vector space. The existence of a basis implies that every representation is a direct sum of one-dimensional (hence irreducible) subrepns.

(2) <u>$\mathcal{A} = k[x]$ and k is algebraically closed</u> :

Irreducible f.d. repns. of $\mathcal{A}$ $\longleftrightarrow$ k

$$\mathbb{1}_\lambda : \left\{ \begin{array}{l} \text{1-dim'l vector space, where} \\ x \text{ acts by the scalar } \lambda. \end{array} \right\} \longleftrightarrow \lambda \in k$$

Proof. - Let $\mathcal{A} \circlearrowright V$ be a f.d. irreducible repn. Since $k = \overline{k}$, $\exists$ an eigenvalue $\lambda \in k$ and an eigenvector $0 \neq v \in V$ for $x \in \mathrm{End}_k(V)$ (i.e. $x \cdot v = \lambda v$).

$\Rightarrow$ $k.v \subset V$ is a non-zero sub-$\mathcal{A}$-repn. of V. By irreducibility, we get $V = k\text{-span of } v$ is 1-dim'l $\square$

<u>Note</u>: n-dimensinal repns. of $k[x]$ $\longleftrightarrow$ $n \times n$ matrices over k

Two repns (n-dim'l) corresponding to matrices X_1, X_2 are isomorphic if and only if $X_2 = g X_1 g^{-1}$ for some $g \in GL_n(k)$.

Jordan canonical form of matrices $\rightsquigarrow$ indecomposable $k[x]$-repns (finite-dim'l)

are given by Jordan blocks:

$$J_\ell(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}_{\ell \times \ell} \quad \longleftrightarrow \quad \begin{array}{l} A \circlearrowright k^\ell \ (\ell\text{-dim'l } k\text{-v.s.}) \\ x \text{ acts by } J_\ell(\lambda). \end{array}$$

→ these are examples of indecomposable repns. which are not irreducible.

→ The statement "every f.d. irred. repn. of $k[x]$ is 1-dim'l" is false if k is not algebraically closed

e.g. $\mathbb{R}[x] \circlearrowright \mathbb{R}^2$ via $x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is irreducible

(but not over $\mathbb{C}$).

(3) Let G be a finite group. $A = k[G]$ = group algebra of G is defined as: $k[G]$ = vector space of functions $f: G \to k$.

Multiplication $$(f_1 * f_2)(g) = \sum_{\substack{g_1, g_2 \in G: \\ g_1 g_2 = g}} f_1(g_1) f_2(g_2) \quad \text{(Convolution product)}$$

$\{\delta_g \in k[G]\}_{(g \in G)}$ - a basis of $k[G]$ $\qquad \delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2}$

($\Rightarrow$ $*$ is an associative operation)

Group representation $G \circlearrowright V$ means we are given a <u>group hom.</u> $\rho : G \to GL(V)$. We can extend it to a ring hom $k[G] \xrightarrow{\rho} End(V)$

by $\rho\left(\sum_{g \in G} c_g \delta_g\right) = \sum_{g \in G} c_g\, \rho(g)$.

Facts about $Rep_{fd}(G;k)$ = category of f.d. repns. of G over k = f.d. repns. of $k[G]$:

[Assuming char(k) does not divide $|G|$] :

- Complete reducibility (Maschke's Thm.). every f.d. G-repn is iso. to a direct sum of irreducible repns. (hence indecomposable $\Rightarrow$ Irreducible).

  Alternately phrased as: $Rep_{fd}(G;k)$ is a <u>semisimple category</u>

- $\#\{\text{f.d. irred. } G\text{-repns}\}/_{\text{iso}}$ = number of conjugacy classes in G.

- $k[G] \cong \bigoplus_{\lambda \in \Lambda_G} End_k(V_\lambda)$ $\quad \Lambda_G$ = set of iso. classes of irred. f.d. repns. of G.

Remark - Many of the results stated for finite groups are true for compact (Lie) groups. e.g. $G = S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$.

$$L^2(S^1; \mathbb{C}) = \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathbb{C}^{(n)} \quad \text{(Fourier theorem)}$$

$$f \mapsto \sum_{n \in \mathbb{Z}} f_n e^{2\pi i n x}$$

- Or more generally - Peter-Weyl theorem.

$$L^2(G) \cong \widehat{\bigoplus_{\lambda \in \Lambda_G}} V_\lambda^{\oplus d_\lambda} \qquad d_\lambda = \dim V_\lambda$$

$\widehat{\oplus}$ signifies infinite sums are allowed, as long as they are L^2-finite.

(4) Algebras given by generators and rel^{n}s.

$\mathcal{A}$: generated by $\{x_i\}_{i\in I}$ subject to rel^{n}s $\{r_j(\underline{x})\}_{j\in J}$ means:

$$\mathcal{A} = k\langle x_i : i\in I\rangle \Big/ \text{two sided ideal gen. by } \{r_j\}_{j\in J}.$$

"free assoc unital alg. generated by x_i $(i\in I)$".

= polynomials in non-commuting variables x_i $(i\in I)$.

e.g. $$U\mathfrak{sl}_2 = k\langle h,e,f\rangle \Big/ \left.\begin{array}{l} ef-fe=h \\ he-eh=2e \\ hf-fh=-2f \end{array}\right\} \text{relations.}$$

Assume $\mathrm{char}(k)=0$. Define L_n = $(n+1)$-dim'l $U\mathfrak{sl}_2$-repn:

$U\mathfrak{sl}_2$-action in a basis $\{v_0, v_1, \dots, v_n\}$:

$$h\,v_j = (n-2j)v_j \quad (0\le j\le n) \;;\quad e.v_j = (n-j+1)v_{j-1}$$

$$f\,v_j = (j+1)v_{j+1} \qquad (v_{-1} = v_{n+1} = 0).$$

Thm. If V is a f.d. irred. repn. of $U\mathfrak{sl}_2$, then $V\cong L_n$ for $n=\dim V-1$.

$\to$ Complete reducibility holds.