Lecture 1

Let G be a finite group, and k be a field.

§1. Representations of G over k. A G-representation (over k) is a k-vector space V together with a linear G-action on V - i.e., a group homomorphism $\alpha : G \longrightarrow \underbrace{GL(V)}$.

↳ group of all linear, invertible maps $V \to V$.

Notation: we often suppress α from our notations and simply write $g \cdot v$ for $\alpha(g)(v)$ $(g \in G, v \in V)$.

A subrepresentation of a G-representation V is a subspace $U \subset V$ such that $g \cdot u \in U$ $\forall g \in G, u \in U$.

If V_1 and V_2 are two G-representations, then

$$\mathrm{Hom}_G(V_1, V_2) := \{ f : V_1 \to V_2 \text{ linear (over } k) \text{ such that } f(g \cdot v_1) = g \cdot (f(v_1)) \ \forall g \in G, v_1 \in V_1 \}$$

$$\subset \mathrm{Hom}_k(V_1, V_2) \quad \text{(all linear maps)}$$

An $f \in \mathrm{Hom}_G(V_1, V_2)$ is called a homomorphism of G-repns or a G-intertwiner.

Remark. - All these notions are exactly the same as the ones we saw for associative algebras last time :

G : finite group $\rightsquigarrow$ $k[G]$ group algebra and

G-repns $=$ $k[G]$-repns via

$$G \circlearrowright V \rightsquigarrow \rho : k[G] \longrightarrow \mathrm{End}_k(V)$$

$$\rho(\delta_g)(v) = g \cdot v \quad \forall g \in G, v \in V.$$

Notation for G acts on V (linearly)

§2. Operations on representations.

(a) Direct Sum : If V_1 and V_2 are two G-repns. then G acts naturally on $V_1 \oplus V_2$ via

$$g \cdot (v_1, v_2) = (g \cdot v_1, g \cdot v_2) \quad \forall g \in G, v_1 \in V_1, v_2 \in V_2$$

(b) Tensor product : Again, let V_1, V_2 be G-repns. We have G-action on $V_1 \otimes V_2$ given by

$$g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2) \quad g \in G, v_j \in V_j \ (j = 1, 2)$$

(c) Dual : If G acts on V, linearly, then the following formula defines a linear G-action on $V^* = \mathrm{Hom}_k(V, k)$.

$$(g\cdot\xi)(v) = \xi(g\cdot v) \qquad \forall\ \xi\in V^*,\ g\in G,\ v\in V.$$

(d) <u>(Internal) Hom</u>. If V_1 and V_2 are G-repns. then G acts on $\mathrm{Hom}_k(V_1,V_2)$ via

$$(g\cdot X)(v) = g\cdot(X(g^{-1}v)) \qquad X: V_1\to V_2,\ g\in G,\ v\in V_1.$$

Remarks. - (i) The operations of tensor product, and dual (b-d) are special for G-repns, and in general, do not exist for arbitrary associative algebra (it needs to be - what is called - a <u>Hopf algebra</u>.)

(ii) Recall - tensor-hom adjointness:

$$\beta: V_1^*\otimes V_2 \longrightarrow \mathrm{Hom}_k(V_1,V_2)$$

$$\beta(\xi_1\otimes v_2): v_1 \longmapsto \xi_1(v_1)v_2$$

Exercise - Show that β is a G-intertwiner.

(iii) For $G \circlearrowright X$, let $X^G := \{x\in X : g\cdot x = x\ \forall g\in G\}$

Exercise - Show that $\mathrm{Hom}_k(V_1,V_2)^G = \mathrm{Hom}_G(V_1,V_2)$.

§3. Indecomposable vs. irreducible representations.

A G-repn. V is said to be <u>indecomposable</u> if

$$V \simeq V_1 \oplus V_2 \implies V_1 = (0) \text{ or } V_2 = (0).$$

We say V is <u>irreducible</u> if $V \neq (0)$ and $(0) \subset V$ is the only proper subrepresentation of V.

Clearly irreducible implies indecomposable. The converse is false, in general.

<u>Example.</u> (1) $G = \mathbb{Z} \longrightarrow GL_2(\mathbb{C})$ indecomposable

$$n \longmapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$ but not irreducible

(2) $G = \mathbb{Z}/n\mathbb{Z}$, $\mathrm{char}(k) = p$ dividing n

$$\mathbb{Z}/n\mathbb{Z} \to GL_2(k)$$ indecomposable

$$x \longmapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$ but not irreducible

§4. <u>Maschke's* Theorem.</u> [G finite, $\mathrm{char}(k) \nmid |G|$]

For ~~e~~every finite dimensional G-representation V and

---

* Heinrich Maschke Oct 24, 1853 – March 1, 1908

a subrepresentation $U \subset V$, there exist a subrepn $W \subset V$ so that $U \oplus W \simeq V$. (as G-repns.)

Proof. - Since every vector space has a basis, and any linearly independent set can be completed to a basis, we have a linear map $\pi : V \to U$ such that $\pi(u) = u$ for each $u \in U$.

Let $P : V \to U$ be defined as the average of π :

$$P(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot (\pi(g^{-1} \cdot v))$$

($|G|$ is invertible in k)

Claim 1. P is a G-intertwiner

(Proof. $$P(g \cdot v) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \cdot (\pi(\underbrace{\sigma^{-1} g}_{h^{-1}} \cdot v))$$

$$= \frac{1}{|G|} \sum_{h \in G} gh \cdot (\pi(h^{-1} \cdot v))$$

$(h = g^{-1}\sigma \Rightarrow \sigma = gh)$

$$= g \cdot P(v) \quad \square \;)$$

<u>Claim 2.</u> $P(u) = u \quad \forall\, u \in U$

(Proof. - $P(u) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}.u)$ since $\pi(x) = x$ $\forall\, x \in U$.

$$= \frac{|G|}{|G|} u = u \quad \square \;)$$

Let $W = \mathrm{Ker}(P) \subset V$ - a subrepn since P is a G-intertwiner.

<u>Claim 3.</u> $V \simeq U \oplus W$.

Proof. - If $x \in U \cap W$, then $x = P(x) = 0$ as $x \in W = \mathrm{Ker}(P)$.
(by claim 2)

Let $v \in V$. Then $v = P(v) + (v - P(v))$ and

$v - P(v) \in \mathrm{Ker}(P) = W$. $\square$

§5. <u>Schur's* Lemma.</u> - Let V and W be G-repns.

and $f : V \to W$ be a (non-zero) G-intertwiner.

(a) If V is irreducible, then f is injective.

(b) If W is irreducible, then f is surjective.

(c) Assume $V = W$ is irred. and finite-dim'l.
Assume k is algebraically closed.

Then $f = \lambda . \mathrm{Id}_V$ for some $\lambda \in k$.

---

* Issai Schur Jan. 10, 1875 - Jan. 10, 1941

Proof. - (a) $\mathrm{Ker}(f) \subset V$ is a subrepn.

(b) $\mathrm{Image}(f) \subset W$ is a subrepn.

(c) Under the assumptions, f has an eigenvalue $\lambda \in k$.

So $\mathrm{Ker}(f - \lambda \cdot \mathrm{Id}) \subset V$ is non-zero subrepn, hence equal to V by irreducibility. $\square$