# Lecture 2

Let $G$ be a finite group and $k$ be an algebraically closed field such that $\mathrm{char}(k)$ does not divide $|G|$.

Recall: <u>Maschke's Theorem</u>: If $V$ is a $G$-representation and $U \subset V$ is a subrepn. then $V \simeq U \oplus W$ as $G$-repns.

<u>Schur's Lemma</u>.- Let $V, W$ be two $G$-repns and $f: V \to W$ a $G$-intertwiner

(1) $V$ irred. $\Rightarrow$ $f = 0$ or $f$ is injective.

(2) $W$ irred. $\Rightarrow$ $f = 0$ or $f$ is surjective.

(3) $V = W$ f.d. irred. $\Rightarrow$ $f = \lambda \cdot \mathrm{Id}_V$ for some $\lambda \in k$.

## §1. <u>Some direct corollaries of these results</u>

(a) Every irreducible $G$-repn is finite-dim'l.

Let $V$ be an irred. $G$-repn and $0 \neq v \in V$ a non-zero vector. Then $\mathrm{Span}\{g.v : g \in G\} \subset V$ is a non-zero subrepn, hence $V = \mathrm{Span}\{g.v : g \in G\}$ implying $\dim V \leq |G| < \infty$.

(b) Every finite-dim'l $G$-repn. is a direct sum of irred repns.

If it is false, we can choose a $G$-repn. of smallest dim. which cannot be written as a direct sum of irred. repns. So, $V$ is not irred itself, and let $U \subsetneq V$ be a proper non-zero repn. By Maschke's theorem $V \simeq U \oplus W$ and $\dim U, \dim W < \dim V$. So, $U$ and $W$ can be written as direct sum of irred. repns. Hence so can $V$ be, contradiction. $\square$

(c) Let $V$ be a finite-dim'l $G$-repn.

Let $\{V_\lambda\}_{\lambda \in P(G)}$ be the set of iso. classes of f.d. irreducible $G$-repns.

( $P(G)$ is just a notation for an indexing set labelling irred. f.d. $G$-repns. We will see later that $P(G)$ is finite - and in 1-1 correspondence with the set of conjugacy classes in $G$.)

Schur's Lemma. - $\operatorname{Hom}_G(V_\lambda, V_\mu) = \begin{cases} k \cdot \mathrm{Id} & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$

Maschke's Theorem. - For a f.d. $G$-repn $V$, we have

$$V \cong \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus d_\lambda(V)}$$

$d_\lambda(V) \in \mathbb{Z}_{\geq 0}$ : multiplicity of $V_\lambda$ in $V$

can be written as, using Schur's lemma:

$$\boxed{d_\lambda(V) = \dim \operatorname{Hom}_G(V, V_\lambda) = \dim \operatorname{Hom}_G(V_\lambda, V)}$$

More "coordinate-free" way to write the previous result is as follows.

Given a finite-dimensional $G$-representation $V$, we have an iso. of $G$-repns:

$$\bigoplus_{\lambda \in P(G)} V_\lambda \otimes \underline{\mathrm{Hom}_G(V_\lambda, V)} \xrightarrow{\sim} V,$$

where $G$-action on the left is on the first tensor factor:

$G \circlearrowright V_\lambda \otimes \underline{\mathrm{Hom}_G(V_\lambda, V)}$ $\qquad g \cdot (v_\lambda \otimes \xi) = (g \cdot v_\lambda) \otimes \xi$.

(often called auxiliary space).

§2. Some examples of representations.

(a) Assume $G$ is abelian. Then every irred. (f.d.) repn. of $G$ is 1-dim'l.

Set of 1-dim'l repns. of a finite group $H$

$$= \mathrm{Hom}_{gp}(H, k^\times)$$

$k^\times$ = multiplicative group $k \setminus \{0\}$.

So, for abelian $G$, $P(G) = \cancel{\{ k_\lambda \,;\, \lambda : G \to k^\times \text{ gp. hom}\}}$

$$= \mathrm{Hom}_{gp}(G, k^\times).$$

(b) Let $X$ be a finite set and assume that $G$ acts on $X$, written as
$$G \times X \to X$$
$$(g, x) \mapsto g\cdot x$$

Then $k[X]$ or $\mathrm{Fun}(X; k)$ the $k$-vector space of all functions $X \to k$, is naturally a $G$-repn. via:

$g \in G$, $f : X \to k$, $x \in X$
$$(g \cdot f)(x) = f(g^{-1}x).$$

e.g. let $x \in X$ and let $\delta_x : X \to k$ be given by $\delta_x(y) = \delta_{x,y} = \begin{cases} 1, & x = y \\ 0, & x \neq y. \end{cases}$

let $g \in G$. Then
$$(g \cdot \delta_x)(y) = \delta_x(g^{-1}y) = \begin{cases} 1 & \text{if } x = g^{-1}y \leftrightarrow y = gx \\ 0 & \text{o/w} \end{cases}$$

i.e. $g \cdot \delta_x = \delta_{gx}$.

Such repns are often called "permutation repns." since matrix repns. of operators from $G$ are permutation matrices (in basis $\{\delta_x : x \in X\}$)

Note: These repns. are not irred (if $|X| \geq 2$), since 1-dim'l subspace of constant functions is a subrepns.

(c) Take $G = S_n$ symmetric group on $n$ letters.

$X = \{1, 2, \ldots, n\}$ with natural $S_n$-action.

$\mathrm{Fun}(X; k)$ is $n$-dim'l and the resulting repn is nothing but $S_n \longrightarrow GL_n(k)$ given by permutation matrices.

In the usual basis $\{\delta_i : 1 \leq i \leq n\}$ ; $\sigma \cdot \delta_i = \delta_{\sigma(i)}$

§3. Decomposition of group algebra. (or regular repn.)

Consider $G$ acting on itself via left multiplication.

$k[G]$ = $k$-vector space of $k$-valued functions on $G$.

The resulting repn. $G \circlearrowright k[G]$ $(\sigma \cdot \delta_h = \delta_{\sigma h})$

is called (left) regular repn.

Theorem. – $$k[G] \cong \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus d_\lambda}$$ where

$$d_\lambda = \dim V_\lambda .$$

Proof. – By Maschke's Theorem $k[G] \cong \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus d_\lambda}$

By Schur's lemma:

$$d_\lambda = \dim \operatorname{Hom}_G(k[G], V_\lambda) \;; \quad \forall \lambda \in P(G).$$

Claim: $\operatorname{Hom}_G(k[G], W) \to W$ is a vector space iso.

$$X \longmapsto X(\delta_e) \qquad (e \in G \text{ unit})$$

for any $G$-repn $W$.

Given this claim, our theorem follows: $d_\lambda = \dim V_\lambda$.

Proof of the claim. Injectivity: if $X : k[G] \to W$ is a $G$-intertwiner such that $X(\delta_e) = 0$. Then

$$X(\delta_g) = X(g\cdot\delta_e) = g\cdot X(\delta_e) = 0 \quad \forall g \in G.$$

i.e. $X = 0$.

Surjectivity: let $w \in W$ and define $k[G] \xrightarrow{Y} W$

$$Y\Big(\sum_{g\in G} c_g \delta_g\Big) = \sum_{g \in G} c_g (g\cdot w).$$

Easy check: $Y$ is a $G$-intertwiner and $Y(\delta_e) = w$. $\square$

§4. Corollaries. — (1) $$\boxed{|G| = \sum_{\lambda \in P(G)} (\dim V_\lambda)^2}$$

(2) $P(G)$ is a finite set.

For $G$ abelian, we get another proof of $|\mathrm{Hom}_{gp}(G, \mathbb{C}^\times)| = |G|$

since each $V_\lambda$ is 1-dim'l.

$$|G| = \sum_{\lambda \in \mathrm{Hom}_{gp}(G, \mathbb{C}^\times)} (\dim k_\lambda)^2 = \sum_{\lambda \in \mathrm{Hom}_{gp}(G, \mathbb{C}^\times)} 1$$

$$= |\mathrm{Hom}_{gp}(G, \mathbb{C}^\times)|.$$