Lecture 3

Recall : G is a finite group and k is a field of $\boxed{\mathrm{char}(k) \nmid |G|}$ and k is algebraically closed.

(char(k) does not divide |G|)

(1) Every finite-dim'l G-repn. splits as a direct sum of irreducible G-repns. (Maschke's Thm)

Notation : Set of iso. classes of irreducible G-repns. $= \{V_\lambda : \lambda \in P(G)\}$

So, if V is a finite-dim'l G-repn, then $V \cong \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus d_\lambda(V)}$,

where $d_\lambda(V) \in \mathbb{Z}_{\geq 0}$ is the multiplicity of V_λ in V.

(2) $$d_\lambda(V) = \dim \mathrm{Hom}_G(V, V_\lambda) = \dim \mathrm{Hom}_G(V_\lambda, V)$$ (Schur's Lemma)

(3) Let $k[G] =$ k-vector space of all functions $G \to k$.

(Left) regular repn : $G \longrightarrow \mathrm{Aut}_{v.s.}(k[G])$

$$(g \cdot f)(x) = f(g^{-1}x) \quad \forall\, g, x \in G;\ f: G \to k \in k[G].$$

Then $$\boxed{k[G] \cong \bigoplus_{\lambda \in P(G)} V_\lambda^{\oplus \dim V_\lambda}}$$

Hence, (a) $|P(G)| < \infty$.

(b) $$|G| = \sum_{\lambda \in P(G)} (\dim V_\lambda)^2$$

§1. Further structures and symmetries of $k[G]$.

(a) <u>$G \times G$ - action.</u> $\quad ((g_1, g_2) \cdot f)(x) = f(g_1^{-1} x g_2)$

here $g_1, g_2, x \in G$ and $f : G \to k$.

Explicitly, in the basis $\{\delta_g : g \in G\}$ of $k[G]$,

$$(g_1, g_2) \cdot \delta_\sigma = \delta_{g_1 \sigma g_2^{-1}} \qquad \forall\, g_1, g_2, \sigma \in G.$$

(b) <u>Convolution product</u> $\quad * : k[G] \times k[G] \to k[G]$

$$(f_1 * f_2)(x) = \sum_{g \in G} f_1(g) f_2(g^{-1}x) = \sum_{\substack{g_1, g_2 \in G \\ \text{s.t. } g_1 g_2 = x}} f_1(g_1) f_2(g_2)$$

Check : (i) $*$ is an associative operation.

(ii) δ_e ($e \in G$ identity element) is the unit for $*$.

(c) For any finite-dim'l G-repn. V, $G \times G$-acts on $\mathrm{End}_k(V)$ via $((g_1, g_2) \cdot X)(v) = g_1(X(g_2^{-1} \cdot v))$

and we have a linear map (actually an alg. hom.)

$$k[G] \longrightarrow \mathrm{End}_k(V)$$

$$\sum_{g \in G} c_g \delta_g \longmapsto \sum_{g \in G} c_g \alpha(g)$$

here $\alpha : G \to GL(V)$ is the underlying group hom.

<u>Lemma.</u> - If $\alpha : G \to GL(V)$ is a group hom, then

$\rho_\alpha : k[G] \longrightarrow \mathrm{End}_k(V)$ is an algebra hom

($k[G]$ under convolution product).

Proof. $\rho_\alpha(\delta_{g_1} * \delta_{g_2}) = \rho_\alpha(\delta_{g_1 g_2}) = \alpha(g_1 g_2)$

$= \alpha(g_1)\alpha(g_2) = \rho_\alpha(\delta_{g_1}) \cdot \rho_\alpha(\delta_{g_2})$ $\square$

§2. <u>Theorem.</u> - $k[G] \xrightarrow{\psi} \bigoplus_{\lambda \in P(G)} \mathrm{End}_k(V_\lambda)$ is an

isomorphism of algebras, and a $G \times G$ - intertwiner.

<u>Proof.</u>- The lemma above shows that ψ is a homomorphism

of algebras.

<u>ψ is injective</u> : Let $\xi = \sum_{g \in G} c_g \delta_g \in \mathrm{Ker}(\psi)$. Then,

for each $\lambda \in P(G)$ - if $\alpha_\lambda : G \to GL(V_\lambda)$ is the action hom-

then $\sum_{g \in G} c_g \, \alpha_\lambda(g) = 0$. Meaning, ξ acts as

the zero operator on each irreducible. By Maschke's theorem,

it must act as zero on every finite-dim'l repn. In particular

ξ acts as zero on the left regular repn. $k[G]$.

So $0 = \xi \cdot \delta_e = \sum_{g \in G} c_g (g \cdot \delta_e) = \xi$, i.e. $\xi = 0$.

Now we obtain that ψ is an iso. from the dimension equality $|G| = \sum_{\lambda \in P(G)} (\dim V_\lambda)^2$.

<u>ψ is a $G \times G$-intertwiner</u>:

$$\psi((g_1, g_2) \cdot \delta_x) = \psi(\delta_{g_1 x g_2^{-1}})$$
$$= \left(\alpha_\lambda (g_1 x g_2^{-1}) \right)_{\lambda \in P(G)}$$
$$= g_1 \cdot \left(\alpha_\lambda(x) \right)_{\lambda \in P(G)} \cdot g_2^{-1} = (g_1, g_2) \cdot \psi(\delta_x). \qquad \square$$

§3. <u>Corollary</u>. - Let $k[G]_{class} = \{ f : G \to k \mid f(g \sigma g^{-1}) = f(\sigma) \ \forall \sigma, g \in G \}$

(functions constant on conjugacy classes)

Then $k[G]_{class} = Z(k[G])$ = center of $k[G]$
$(:= \{ x \in k[G] : a * x = x * a \ \forall a \in k[G] \})$

$= k[G]^G \leftarrow$ G-invariants, where G action is via conjugation.

and ψ sets up an iso. $\boxed{\psi : k[G]_{class} \cong \bigoplus_{\lambda \in P(G)} k \cdot Id_{V_\lambda}}$

Proof. We consider the G-action via conjugation – i.e. restricting the $G\times G$-action on $k[G]$ and $\mathrm{End}_k(V_\lambda)$ to the diagonal copy of $\begin{matrix} G \to G\times G \\ g \mapsto (g,g) \end{matrix}$.

As ψ is $G\times G$-intertwiner, it is also G-intertwiner. Taking G-invariants, we get

$$\psi : k[G]^G \cong \bigoplus_{\lambda\in P(G)} \mathrm{End}_k(V_\lambda)^G$$

by Schur's lemma.

$$= \bigoplus_{\lambda\in P(G)} \mathrm{End}_G(V_\lambda) = \bigoplus_{\lambda\in P(G)} k\cdot \mathrm{Id}_{V_\lambda}$$

Now $k[G]^G = \{ f : G\to k \text{ s.t. } f(g\sigma g^{-1}) = f(\sigma)\ \forall g,\sigma\in G\}$

$= k[G]_{class}$

(Exercise – it is also equal to $Z(k[G])$.) $\square$

Summary

$$\begin{array}{ccc} \psi : k[G] & \xrightarrow{\cong} & \bigoplus_{\lambda\in P(G)} \mathrm{End}_k(V_\lambda) \\ \cup & & \cup \\ k[G]_{class} & \xrightarrow{\cong} & \bigoplus_{\lambda\in P(G)} k\cdot \mathrm{Id}_{V_\lambda} \end{array}$$

Hence $\boxed{|\text{Conjugacy classes in } G| = |P(G)|}$

§4. Character of a group representation. – (Frobenius 1896)

Let $\alpha : G \to GL(V)$ be a finite-dim'l representation of G.

Define $\chi_V : G \to k$ by $\chi_V(g) = \text{Trace}(\alpha(g))$

<u>Theorem.</u> – (1) $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$ (0) $\chi_V \in k[G]_{\text{class}}$

(2) $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$

(3) $\chi_{V^*}(g) = \chi_V(g^{-1})$

(4) Let $B : k[G] \times k[G] \longrightarrow k[G]$ denote the bilinear form $B(f_1, f_2) = \dfrac{1}{|G|} \sum_{g \in G} f_1(g^{-1}) f_2(g)$

(Note: B is a symmetric bilinear form). Then

$$B(\chi_{V_1}, \chi_{V_2}) = \dim \operatorname{Hom}_G(V_1, V_2)$$

<u>Proof</u>: (0) since $Tr(ABA^{-1}) = Tr(B)$ for any B and invertible A

(1) since $Tr \left[\begin{array}{c|c} A_1 & * \\ \hline 0 & A_2 \end{array}\right] = Tr(A_1) + Tr(A_2)$.

(2) since $Tr(A \otimes B) = Tr(A) \cdot Tr(B)$

(proof – assume $A : V \to V$, $B : W \to W$, V and W are k-vector spaces of dim m and n resp.

Ferdinand Georg Frobenius Oct. 26, 1849 – Aug. 3, 1917

Pick bases $\{v_1, \dots, v_m\}$ of V, $\{w_1, \dots, w_n\}$ of W and write A and B as matrices

$$A v_j = \sum_{i=1}^{m} a_{ij} v_i$$

$$B w_t = \sum_{s=1}^{n} b_{st} w_s$$

so that $\mathrm{Tr}(A) = \sum_{i=1}^{m} a_{ii}$; $\mathrm{Tr}(B) = \sum_{s=1}^{n} b_{ss}$. Now,

$$A \otimes B : v_j \otimes w_t \mapsto \sum_{\substack{1 \le i \le m \\ 1 \le s \le n}} a_{ij} b_{st} v_i \otimes w_s ,$$

so sum of its diagonal entries : $\mathrm{Tr}(A \otimes B) = \sum_{j,t} a_{jj} b_{tt} = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$ $\square$)

(3) Let V be a finite-dim'l G-repn ; $\{v_1, \dots, v_n\}$ a basis of V and $\{\eta_1, \dots, \eta_n\}$ the dual basis of V^* ($\eta_i(v_j) = \delta_{ij}$).

$$\chi_{V^*}(g) = \sum_{i=1}^{n} \text{Coefficient of } \eta_i \text{ in } g \cdot \eta_i$$

$$= \sum_{i=1}^{n} (g \cdot \eta_i)(v_i) = \sum_{i=1}^{n} \eta_i (g^{-1} \cdot v_i)$$

$$= \sum_{i=1}^{n} \text{coeff. of } v_i \text{ in } g^{-1} \cdot v_i = \text{Trace of } g^{-1} \text{ acting on } V$$

$$= \chi_V(g^{-1}).$$

(4) $$B(\chi_{V_1}, \chi_{V_2}) = \frac{1}{|G|} \sum_{g \in G} \chi_{V_1}(g^{-1})\, \chi_{V_2}(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{V_1^* \otimes V_2}(g) \quad \text{by (2) and (3) above}$$

$$= \text{Trace of } \frac{1}{|G|} \sum_{g \in G} g \text{ acting on } V_1^* \otimes V_2 = \operatorname{Hom}_k(V_1, V_2)$$

Ex. For any f.d. G-repn U, the averaging operator

$$P = \frac{1}{|G|} \sum_{g \in G} g : U \to U \text{ is a projection onto}$$

$U^G = \{ u \in U : \sigma . u = u \;\; \forall \sigma \in G \}$. Hence $\operatorname{Tr}_U(P) = \dim U^G$.

$$= \dim \operatorname{Hom}_k(V_1, V_2)^G = \dim \operatorname{Hom}_G(V_1, V_2) \qquad \square$$

§5. Corollary: $\{ \chi_{V_\lambda} : \lambda \in P(G) \}$ is an orthonormal basis of $k[G]_{class}$ under the symmetric bilinear form B introduced above.