Lecture 4

Recall : G is a finite group. k is an algebraically closed field and $\mathrm{char}(k)$ does not divide $|G|$.

$k[G]$ = (convolution) algebra of all k-valued functions on G.

(Thm §2 of Lecture 3) : $\psi : k[G] \xrightarrow{\sim} \bigoplus_{\lambda \in P(G)} \mathrm{End}_k(V_\lambda)$ iso. of algebras

Notation: $\{\alpha_\lambda : G \to GL(V_\lambda)\}_{\lambda \in P(G)}$ is the set of iso. classes of irred. G-repns.

$$\psi(\delta_g) = (\alpha_\lambda(g))_{\lambda \in P(G)} \quad \text{for each } g \in G.$$

$$k[G]_{class} = \{ f : G \to k \mid f(g\sigma g^{-1}) = f(\sigma) \ \forall g, \sigma \in G \} \subset k[G].$$

Then,

$$k[G]_{class} = Z(k[G]) \quad \text{center of } k[G]$$

$$= k[G]^{G\text{-conj}} \quad G\text{-invariants under the conjugation action.}$$

and ψ gives an iso.

$$k[G]_{class} \xrightarrow{\sim} \bigoplus_{\lambda \in P(G)} k \cdot \mathrm{Id}_{V_\lambda}$$

scalar matrices $\left\{ \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} : x \in k \right\} \subset \mathrm{End}_k(V_\lambda)$

§1. Bilinear form on $k[G]$, characters and orthogonality :

$$B(f_1, f_2) = \frac{1}{|G|}(f_1 * f_2)(e) = \frac{1}{|G|} \sum_{g \in G} f_1(g^{-1}) f_2(g)$$

$B : k[G] \times k[G] \to k$ is a symmetric bilinear form.

For V, a finite dim'l G-representation, let $\chi_V \in k[G]_{class}$ be given by $\chi_V(g) = Tr_V(g) =$ trace of g acting on V.

Then, (see Lecture 3, Thm §4):

$$\boxed{B(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V, W)}$$

In particular, $\{\chi_{V_\lambda}\}_{\lambda \in P(G)} \subset k[G]_{class}$ is a set of orthonormal vectors, hence they are linearly independent. As $|P(G)| =$ number of conjugacy classes in G $= \dim k[G]_{class}$, we get that $\{\chi_{V_\lambda}\}$ is a basis of $k[G]_{class}$.

§2. Example, $G = S_3$. Permutations of $\{1,2,3\}$.

Conjugacy classes $C_1 = \{id\}$; $C_2 = \{(12), (23), (13)\}$

$C_3 = \{(123), (132)\}$

$\mathbb{1}$: Trivial repn. $S_3 \to \mathbb{C}^\times$, $\sigma \mapsto 1 \ \forall \sigma \in S_3$ $\qquad \chi_{\mathbb{1}}(\sigma) = 1 \ \forall \sigma \in G$.

ε : sign repn $S_3 \to \mathbb{C}^\times$, $(12), (13), (23) \mapsto (-1)$ $\qquad \chi_\varepsilon(\sigma) = 1$ for $\sigma \in C_1 \cup C_3$, -1 for $\sigma \in C_2$

Character table of S_3

	$\chi_{\mathbb{1}}$	χ_ε	χ_V
C_1	1	1	2
C_2	1	-1	x
C_3	1	1	y

(We know there are 3 irred repns = # Conjugacy classes)

$V_1 = \mathbb{1}$ trivial
$V_2 = \varepsilon$ sign. By $6 = 1^2 + 1^2 + 2^2$
$V_3 = V$ is 2-dim'l

Orthogonality $\displaystyle \frac{1}{6} \sum_{j=1}^{3} |C_j| \, \chi_{V_k}(C_j) \, \chi_{V_\ell}(C_j) = \delta_{k,\ell}$

e.g. $B(\chi_{\mathbb{1}}, \chi_\varepsilon) = \frac{1}{6}\left(1 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot (-1) + 2 \cdot 1 \cdot 1\right) = 0$

$B(\chi_{\mathbb{1}}, \chi_{\mathbb{1}}) = \frac{1}{6}(1 + 3 + 2) = 1.$

Ex. Use orthogonality rel^{n}s $B(\chi_{\mathbb{1}}, \chi_V) = B(\chi_\varepsilon, \chi_V) = 0$ to get $x = 0$, $y = -1$.

§3. Change of basis : $k[G]_{\text{class}}$ has two natural bases

$\{\delta_C : C \subset G \text{ a conjugacy class}\}$ and $\{\chi_{V_\lambda} = \chi_\lambda\}_{\lambda \in P(G)}$

$$\delta_C(g) = \begin{cases} 1 & \text{if } g \in C \\ 0 & \text{otherwise} \end{cases}$$

Prop.- $$\delta_C = \sum_{\lambda \in P(G)} \frac{|C|}{|G|} \chi_\lambda(\sigma^{-1}) \, \chi_\lambda$$

for some (or any) $\sigma \in C$.

Proof. - If $\delta_C = \sum_{\lambda \in P(G)} a_\lambda \cdot \chi_\lambda$ where $a_\lambda \in k$, then

$$B(\chi_\lambda, \delta_C) = a_\lambda = \frac{1}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1}) \boxed{\delta_C(g)}$$

$$\boxed{\delta_C(g)} = \begin{cases} 1 & \text{for } g \in C \\ 0 & \text{for } g \notin C \end{cases}$$

Since χ_λ is constant on conjugacy classes, and $x \sim y \iff x^{-1} \sim y^{-1}$, we get (pick any $\sigma \in C$)

$$a_\lambda = \frac{1}{|G|} \sum_{g \in C} \chi_\lambda(g^{-1}) = \frac{|C|}{|G|} \chi_\lambda(\sigma^{-1})$$ and we are done $\square$

<u>Cor. (Orthogonality of rows of the character table)</u>:

$$\sum_{\lambda \in P(G)} \chi_\lambda(g_1^{-1}) \chi_\lambda(g_2) = \begin{cases} \frac{|G|}{|C|} & \text{if } g_1 \text{ and } g_2 \text{ are conjugate} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As $\delta_C = \frac{|C|}{|G|} \sum_{\lambda \in P(G)} \chi_\lambda(\sigma^{-1}) \chi_\lambda$, we get

($C \subset G$ conjugacy class of g_1, $\sigma = g_1$)

$$\underbrace{\delta_C(g_2)}_{\begin{cases} 1 & \text{if } g_2 \in C \\ 0 & \text{o/w} \end{cases}} = \frac{|C|}{|G|} \sum_{\lambda \in P(G)} \chi_\lambda(g_1^{-1}) \chi_\lambda(g_2)$$

$\square$

e.g. character table of S_3 (Note: in this case σ is conjugate to σ^{-1})

		χ_1	χ_2	χ_3
1	C_1	1	1	2
3	C_2	1	-1	0
2	C_3	1	1	-1

↑ number of elts per conjugacy class

C_1 vs C_2:

$$1(1)+1(-1)+2(0)=0.$$

C_2 vs C_2:

$$1^2+(-1)^2+0^2 = 2 = \frac{6}{3} = \frac{|S_3|}{|C_2|}.$$

§4. Projection operators. "generalized averages"

<u>Theorem.</u> Let $\lambda \in P(G)$. Then, for any f.d. G-repn $\alpha: G \to GL(V)$

$$P_\lambda = \frac{\dim(V_\lambda)}{|G|} \sum_{g\in G} \chi_\lambda(g^{-1})\,\alpha(g) : V \to V$$

is projection onto V_λ "isotypical component of V"

(if $V \cong \bigoplus_\mu V_\mu^{\oplus d_\mu(V)}$, then $U_\lambda = V_\lambda^{\oplus d_\lambda(V)}$ is called the V_λ isotypical component)

<u>Proof.</u>- We claim that for any $f \in k[G]_{\text{class}}$,

$$\pi_f = \sum_{g\in G} f(g^{-1})\,\alpha(g) : V \to V \text{ is a } G\text{-intertwiner}$$

(pf. $\pi_f(\sigma . v) = \sum_{g\in G} f(g^{-1})\,\alpha(g\sigma)(v)$ set $\sigma h = g\sigma$ so, $g = \sigma h \sigma^{-1}$, $g^{-1} = \sigma h^{-1}\sigma^{-1}$

$$= \sum_{h\in G} f(\sigma h^{-1}\sigma^{-1})\,\alpha(\sigma h)(v)$$

$$= \alpha(\sigma)\left(\sum_{h\in G} f(h^{-1})\,\alpha(h)(v)\right)$$

where we used $f(\sigma x \sigma^{-1}) = f(\sigma)$ and $\alpha(\sigma h) = \alpha(\sigma)\alpha(h)$.

So, $\pi_f(\sigma \cdot v) = \sigma \cdot \pi_f(v)$ as claimed.)

Now we take $f = \chi_{V_\lambda}$; $P_{\lambda,\alpha} = \dfrac{\dim V_\lambda}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1}) \cdot \alpha(g)$.

The theorem follows from the second claim:

<u>Claim 2.</u> $P_{\lambda, \alpha_\mu} = \begin{cases} Id_{V_\lambda} & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda. \end{cases}$ $\left(\alpha_\mu : G \to GL(V_\mu)\right.$ irred. repn. for $\left.\mu \in P(G)\right)$

As $P_\lambda : V_\mu \to V_\mu$ is a G-intertwiner, by Schur's lemma $P_\lambda = x \cdot Id_{V_\mu}$ for some $x \in k$. To compute x, we take trace:

$$x \cdot \dim V_\mu = \text{Trace of } P_\lambda \text{ acting on } V_\mu$$

$$= \frac{\dim V_\lambda}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1}) \chi_\mu(g)$$

$$= (\dim V_\lambda) \cdot B(\chi_\lambda, \chi_\mu) = \dim V_\lambda \cdot \delta_{\lambda,\mu}.$$

$\Rightarrow x = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$ as claimed. $\square$