Lecture 5

Let G be a finite group and k an algebraically closed field, such that $\mathrm{Char}(k)$ does not divide $|G|$.

Let $\mathrm{Irr}_{f.d.}(G;k)$ = set of iso. classes of G-representations over k.

$$= \{\alpha_\lambda : G \to GL(V_\lambda)\}_{\lambda \in P(G)} \quad \text{(Notation).}$$

Fundamental Theorem of repn. th. of finite groups :

$$\begin{array}{ccc} \Psi : k[G] & \xrightarrow{\sim} & \bigoplus_{\lambda \in P(G)} \mathrm{End}(V_\lambda) \quad \text{iso. of algebras} \\ \cup & & \cup \\ k[G]_{\text{class}} & \xrightarrow{\sim} & \bigoplus_{\lambda \in P(G)} k \cdot \mathrm{Id}_{V_\lambda} \end{array}$$

Structures on $k[G]$: (i) Convolution product

$$(f_1 * f_2)(\sigma) = \sum_{g \in G} f_1(\sigma g^{-1}) f_2(g)$$

(ii) Bilinear form $\quad B(f_1, f_2) = \frac{1}{|G|}(f_1 * f_2)(e) = \frac{1}{|G|}\sum_{g \in G} f_1(g^{-1}) f_2(g)$

Last time we showed :

$$B(\chi_V, \chi_W) = \dim_{k\text{-v.s.}} \mathrm{Hom}_G(V, W), \quad \text{where}$$

$$\chi_V : G \to k \qquad (V : \text{a f.d. } G\text{-repn}) \in k[G]_{\text{class}}.$$
$$g \mapsto \text{Trace of } g \text{ acting on } V$$

Thus we have two bases for $k[G]_{\text{class}}$

$$\{\delta_C : C \subset G \text{ a conjugacy class}\}, \quad \delta_C(g) = \begin{cases} 1 & \text{if } g \in C \\ 0 & \text{if } g \notin C \end{cases}$$

$$\{\chi_\lambda = \chi_{V_\lambda} : \lambda \in P(G)\} \qquad \chi_\lambda(g) = \operatorname{Tr}(\alpha_\lambda(g))$$

(recall: by Schur's lemma, $B(\chi_\lambda, \chi_\mu) = \delta_{\lambda\mu}$.)

§1. Lemma – $$\delta_C = \frac{|C|}{|G|} \sum_{\lambda \in P(G)} \chi_\lambda(g^{-1})\, \chi_\lambda \qquad (g \in C)$$

Proof – If $\delta_C = \sum_{\lambda \in P(G)} x_\lambda \cdot \chi_{V_\lambda}$, then by orthonormality of characters, we have

$$\begin{aligned} x_\lambda &= B(\chi_\lambda, \delta_C) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1})\, \delta_C(g) \end{aligned}$$

Since $\delta_C(g) = \begin{cases} 1 & \text{if } g \in C \\ 0 & \text{o/w} \end{cases}$, and $g_1, g_2 \in C \Rightarrow g_1 = \sigma g_2 \sigma^{-1}$
$\Rightarrow g_1^{-1} = \sigma g_2^{-1} \sigma^{-1}$
$\Rightarrow \chi_\lambda(g_1^{-1}) = \chi_\lambda(g_2^{-1})$

we get $x_\lambda = \dfrac{|C|}{|G|} \chi_\lambda(g^{-1})$ for (any) $g \in C$. □

Cor. Orthogonality of rows of character table.

$$\sum_{\lambda \in P(G)} \chi_\lambda(g^{-1})\, \chi_\lambda(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ and } g \text{ are not conjugate} \\ \dfrac{|G|}{|C|} & \text{if they are} \end{cases}$$

§2. Orthogonality of matrix coefficients.-

Definition.- If V is a repn. of G, $v \in V$ and $\xi \in V^*$, then define $c_{v,\xi} : G \to k$ by

$$g \mapsto \xi(g^{-1}.v)$$

Thus, "matrix coefficient" map is a linear map $V \otimes V^* \xrightarrow{M} k[G]$

$$v \otimes \xi \longmapsto c_{v,\xi}.$$

Ex: Check M is a $G \times G$-intertwiner

The following result gives another basis of $k[G]$:

let $n_\lambda = \dim(V_\lambda)$, $\{v_1^{(\lambda)}, \ldots, v_{n_\lambda}^{(\lambda)}\}$ a basis of V_λ

$\{\eta_1^{(\lambda)}, \ldots, \eta_{n_\lambda}^{(\lambda)}\}$ the dual basis of V_λ^*.

$$c_{ij}^{(\lambda)} = \text{matrix coefficient of } v_i^{(\lambda)} \otimes \eta_j^{(\lambda)} \quad (1 \leq i,j \leq n_\lambda)$$

$$c_{ij}^{(\lambda)}(g) = \eta_j^{(\lambda)}(g^{-1}.v_i^{(\lambda)})$$

Prop. $$B(c_{ij}^{(\lambda)}, c_{st}^{(\mu)}) = \delta_{\lambda\mu} \cdot \delta_{it}\,\delta_{js} \cdot \frac{1}{n_\lambda}.$$

Proof - $$B(c_{ij}^{(\lambda)}, c_{st}^{(\mu)}) = \frac{1}{|G|} \sum_{g \in G} c_{ij}^{(\lambda)}(g^{-1})\, c_{st}^{(\mu)}(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \eta_j^{(\lambda)}(g.v_i^{(\lambda)})\ \eta_t^{(\mu)}(g^{-1}.v_s^{(\mu)})$$

$$= \frac{1}{|G|} \sum_{g \in G} \eta_j^{(\lambda)}(g.v_i^{(\lambda)})\ (g.\eta_t^{(\mu)})(v_s^{(\mu)})$$

Hence, $B(c_{ij}^{(\lambda)}, c_{st}^{(\mu)})$ = evaluation of $\eta_j^{(\lambda)} \otimes v_s^{(\mu)}$ on the vector $P(v_i^{(\lambda)} \otimes \eta_t^{(\mu)})$

recall $P = \frac{1}{|G|} \sum_{g \in G} g$ is our averaging operator.

$$V_\lambda \otimes V_\mu^* \xrightarrow{P} V_\lambda \otimes V_\mu^* \xrightarrow[\eta_j^{(\lambda)} \otimes v_s^{(\mu)} \in V_\lambda^* \otimes V_\mu]{\text{evaluate at}} k.$$

Since P is projection onto trivial G-isotypical part,

$$P(v_i^{(\lambda)} \otimes \eta_t^{(\mu)}) \in (V_\lambda \otimes V_\mu^*)^G = \mathrm{Hom}_G(V_\mu, V_\lambda) = \{0\} \text{ if } \lambda \neq \mu.$$

Assuming $\lambda = \mu$, $P(v_i^{(\lambda)} \otimes \eta_t^{(\lambda)}) \in \underbrace{\mathrm{Hom}_G(V_\lambda, V_\lambda)}_{\text{1-dim'l}} \cong (V_\lambda \otimes V_\lambda^*)^G$.

$$\mathrm{Id}_{V_\lambda} \longleftrightarrow \sum_{\ell=1}^{n_\lambda} v_\ell^{(\lambda)} \otimes \eta_\ell^{(\lambda)}$$

i.e. $\exists\, x \in k$ s.t.

$$P(v_i^{(\lambda)} \otimes \eta_t^{(\lambda)}) = x \cdot \sum_{\ell=1}^{n_\lambda} v_\ell^{(\lambda)} \otimes \eta_\ell^{(\lambda)}.$$

$\Rightarrow$ evaluation at $\eta_j^{(\lambda)} \otimes v_s^{(\mu)}$ $= x \cdot \delta_{js}$. To find the value of x, we apply the "evaluation map" $\delta_{it} = x \cdot n_\lambda$. $\square$

$$V_\lambda \otimes V_\lambda^* \to k \qquad \text{(exercise: it is a } G\text{-intertwiner)}$$

$$v \otimes \xi \mapsto \xi(v)$$