Lecture 8

Recall the notations.– S_n = symmetric group on n letters.

$$P(n) = \left\{ \lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_\ell > 0) \;\middle|\; \sum_{j=1}^{\ell} \lambda_j = n \right\} \quad (\text{partitions of } n)$$

Conjugacy classes in S_n $= \{ C(\mu) : \mu \in P(n) \}$

Irreducible finite dim'l repns of S_n $= \{ V_\lambda : \lambda \in P(n) \}$

For $\mu \in P(n)$ (also written as $\mu \vdash n$: read: "μ is a partition of n)

$$|C(\mu)| = \frac{n!}{z(\mu)} \quad ; \quad z(\mu) = \prod_{i \geq 1} i^{k_i} \, k_i!$$

$$(k_i = \# \{ j \mid \mu_j = i \})$$

Last time we introduced a family of "partition / permutation / flag repns." of S_n: $\{ U_\lambda : \lambda \in P(n) \}$ as follows:

- $X(\lambda) := \left\{ (I_1, \ldots, I_\ell) \;\middle|\; \begin{array}{l} [n] = I_1 \sqcup I_2 \cdots \sqcup I_\ell \\ |I_j| = \lambda_j \ \forall j \end{array} \right\}$

  Note: $|X(\lambda)| = \dfrac{n!}{\lambda_1! \cdots \lambda_\ell!}$, S_n acts on $X(\lambda)$ and the action is <u>transitive</u> (i.e., there is only one S_n-orbit)

- U_λ = $\mathbb{C}$-v.s. of Functions $X(\lambda) \to \mathbb{C}$

Let $\chi_\lambda \in (\mathbb{C} S_n)_{\text{class}}$ denote the character of U_λ.

We computed this character in the previous lecture (§3 of Lecture 7)

$$\chi_\lambda(\mu) := \chi_\lambda(w_\mu) \qquad (w_\mu \in C(\mu) \;;\; \lambda, \mu \vdash n)$$

$$= \text{Trace of } w_\mu \text{ acting on } V_\lambda$$

$$= \#\{ x \in X(\lambda) : w_\mu \cdot x = x \}$$

[Take $w_\mu = (1\,2 \cdots \mu_1)(\mu_1+1 \cdots \mu_1+\mu_2) \cdots$ and let

$J_s = \{\mu_1 + \dots + \mu_{s-1} + 1, \dots, \mu_1 + \dots + \mu_s\}$ $(s \geq 1)$]

$$\chi_\lambda(\mu) = \#\{ (I_1, \dots, I_\ell) \in X(\lambda) : \text{each } I_j \text{ is a union of (some of) } J_1, J_2, \dots \}$$

$$= \text{coefficient of } x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell} \text{ in } \prod_{j \geq 1} (x_1^{\mu_j} + \dots + x_N^{\mu_j}) .$$

(here $N \geq n$)

§1. Keep $N \geq n$, $N =$ number of variables.

Let $\Lambda_n = \mathbb{Z}[x_1, \dots, x_N]^{S_N}_{\text{degree}=n}$. The result proved above can be rephrased as:

$$\boxed{p_\mu = \sum_{\lambda \vdash n} \chi_\lambda(\mu)\, m_\lambda}$$

where for $\underline{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^n$,

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$$

and we define:

- Monomial basis: $m_\lambda = \sum_{\alpha \in S_N \cdot \lambda} x^\alpha$
- Power Sum: $p_r = x_1^r + x_2^r + \dots + x_N^r$ $(\forall\, r \geq 1)$

and $p_\mu = p_{\mu_1} \cdots p_{\mu_K}$ if $\mu = (\mu_1 \geq \dots \geq \mu_K)$

Lemma. - (i) $\{m_\lambda\}_{\lambda \vdash n}$ is a $\mathbb{Z}$-basis of Λ_n.

(ii) $\{p_\lambda\}_{\lambda \vdash n}$ is a $\mathbb{Q}$-basis of $\Lambda_{n,\mathbb{Q}} = \Lambda_n \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Proof - (i) If $f \in \Lambda_n$, then $f = \sum_\alpha c_\alpha x^\alpha \quad (c_\alpha \in \mathbb{Z})$

and $w \cdot f = f$ means $c_\alpha = c_{w \cdot \alpha} \quad (w \in S_N)$. Hence,

$f = \sum_{\lambda \vdash n} c_\lambda m_\lambda$ proving that $\{m_\lambda\}_{\lambda \vdash n}$ span (over $\mathbb{Z}$) Λ_n.

By comparing the multi-degrees, it is easy to see that $\{m_\lambda\}$ are linearly independent over $\mathbb{Z}$.

(ii) Consider the lexicographic ordering on the set $\mathcal{P}(n)$. That is,

$\lambda \geq \mu$ means $\exists\, i$ s.t. $\lambda_1 = \mu_1, \ldots, \lambda_{i-1} = \mu_{i-1}$ & $\lambda_i > \mu_i$
$(1 \leq i \leq \ell(\lambda)+1)$.

From the combinatorial description of $\chi_\lambda(\mu)$ given in the line before §1 above, it is clear that $\chi_\lambda(\mu) \neq 0 \Rightarrow \mu \leq \lambda$.

Hence the "change of basis" matrix $\{\chi_\lambda(\mu)\}$ is triangular with non-zero entries on the diagonal $(\chi_\lambda(\lambda) \neq 0)$.

$$(p_\mu) = (\chi_\lambda(\mu)) \cdot (m_\lambda)$$

$\mathbb{Z}$-basis

triangular, invertible (over $\mathbb{Q}$) matrix

$\Rightarrow \{p_\mu\}_{\mu \vdash n}$ form a basis of the $\mathbb{Q}$-vector space $\Lambda_n \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. $\square$

§2. Frobenius' character map. $(\mathbb{C}S_n)_{class} \xrightarrow{\mathcal{F}} \Lambda_{n,\mathbb{C}}$

is a vector space isomorphism given by $\boxed{\mathcal{F}(\delta_{C(\mu)}) = \dfrac{p_\mu}{z(\mu)}}$ $\forall \mu \vdash n$.

The non-degenerate, symmetric bilinear form on $(\mathbb{C}S_n)_{class}$ is given by

$$(f_1, f_2) = \frac{1}{n!} \sum_{w \in S_n} f_1(w^{-1}) f_2(w) = \frac{1}{n!} \sum_{w \in S_n} f_1(w) f_2(w)$$

[For S_n; w and w^{-1} are in the same conjugacy class]

$$= \frac{1}{n!} \sum_{\lambda \vdash n} |C(\lambda)|\, f_1(\lambda)\, f_2(\lambda)$$

[$f(\lambda) = f(w)$ for any $w \in C(\lambda)$]

$$= \sum_{\lambda \vdash n} \frac{f_1(\lambda) f_2(\lambda)}{z(\lambda)}$$

$$\Rightarrow \quad (\delta_{C(\lambda)}, \delta_{C(\mu)}) = \frac{\delta_{\lambda,\mu}}{z(\lambda)}$$

This bilinear form transported to $\Lambda_{n,\mathbb{C}}$ via the iso. $\mathcal{F} : (\mathbb{C}S_n)_{class} \to \Lambda_{n,\mathbb{C}}$

becomes: $\boxed{(p_\lambda, p_\mu) = \delta_{\lambda\mu} \cdot z(\lambda)}$ "Green inner product"

Hence, Ch becomes an isometry when $\Lambda_{n;\mathbb{C}}$ is endowed with the Green inner product.

§3. Computation of the inner product on Λ_n

General remark.- Let E be a f.d. vector space (say, over $\mathbb{Q}$) and let $B : E \times E \to \mathbb{Q}$ be a non-degenerate, bilinear form. Thus, B defines an isomorphism $\beta : E^* \xrightarrow{\sim} E$ by:

$$B(\beta(\xi), v) = \xi(v) \quad \forall\, \xi \in E^*,\ v \in E.$$

The canonical tensor of B is defined as:

$$\begin{array}{ccccc} (E \otimes E)^* & \simeq & E^* \otimes E^* & \simeq & E \otimes E \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ B & - - - - - - - - & & \dashrightarrow & \Omega_B \end{array} \quad (\text{canonical tensor of } B)$$

Concretely, pick a basis $\{v_i\}$ of E and let $\{w_i\}$ be the basis of E dual to $\{v_i\}$ w.r.t. B (i.e., $B(v_i, w_j) = \delta_{ij}$). Then

$$\Omega_B = \sum_i v_i \otimes w_i$$

Prop. The canonical tensor of the inner product on $\Lambda_{n,\mathbb{Q}}$ is given by (degree n term of) $\displaystyle\prod_{i,j} \frac{1}{1 - x_i y_j}$.

(Here, we view $\Lambda_n \otimes \Lambda_n$ as polynomials in $\{x_1, \dots, x_N\} \cup \{y_1, \dots, y_N\}$ symmetric in $\underline{x}$ and $\underline{y}$ separately; with degree in $\underline{x}$ = deg in $\underline{y} = n$

$$\Lambda_n \otimes \Lambda_n = \mathbb{Z}[x_1, \dots, x_N ; y_1, \dots, y_N]^{S_N \times S_N}_{\text{degree } n} \; .)$$

Proof – According to the formula $\underline{(p_\lambda, p_\mu) = \delta_{\lambda\mu} \cdot z(\mu)}$, the dual basis to the basis $\{p_\lambda\}_{\lambda \vdash n}$ is $\left\{ \frac{p_\lambda}{z(\lambda)} \right\}_{\lambda \vdash n}$. That is, the canonical tensor of this bilinear form is

$$\Omega_n(x;y) = \sum_{\lambda \vdash n} \frac{p_\lambda(x)\, p_\lambda(y)}{z(\lambda)} \qquad \left[\begin{array}{l} x = (x_1, \dots, x_N) \\ y = (y_1, \dots, y_N) \end{array} \right]$$

Consider the following sum: $\Omega(x;y) = \sum_{n \geq 0} \Omega_n(x;y)$ $\quad [\Omega_0 = 1]$

$$= 1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash n} \frac{p_\lambda(x)\, p_\lambda(y)}{z(\lambda)}$$

Rewrite this sum using the other description of partitions:

$$\mathcal{P}(n) = \left\{ \underline{\ell} : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0} \ : \ \sum_{j \geq 1} j \ell_j = n \right\}$$

using:

$$\left[\begin{array}{l} z(\underline{\ell}) = \prod_{j \geq 1} j^{\ell_j} \cdot \ell_j! \\ p_\lambda = \prod_{j \geq 1} p_j^{\ell_j} \end{array} \right]$$

$$\Omega(x,y) = \sum_{\substack{\underline{\ell} : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0} \\ \text{of finite support} \\ (|\{j : \ell_j \neq 0\}| < \infty)}} \prod_{j \geq 1} \left(\frac{p_j(x)\, p_j(y)}{j} \right)^{\ell_j} \frac{1}{\ell_j!}$$

$$= \prod_{j \geq 1} \sum_{\ell = 0}^{\infty} \left(\frac{p_j(x)\, p_j(y)}{j} \right)^{\ell} \cdot \frac{1}{\ell!}$$

$$= \prod_{j \geq 1} \exp\left(\frac{p_j(x)\, p_j(y)}{j} \right) = \exp\left(\sum_{j=1}^{\infty} \frac{p_j(x)\, p_j(y)}{j} \right)$$

Now
$$\sum_{r=1}^{\infty} \frac{p_r(x)\,p_r(y)}{r} = \sum_{i,j} \sum_{r=1}^{\infty} \frac{(x_i y_j)^r}{r}$$
$$= \sum_{i,j} -\log(1 - x_i y_j) = \log.\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right)$$

$$\Rightarrow \Omega(x;y) = \prod_{i,j} \frac{1}{1-x_i y_j} \qquad \square$$

§4. Meaning of Prop. §3.

Assume we have a basis $\{u_\lambda\}_{\lambda \vdash n}$ of $\Lambda_{n;\mathbb{Q}}$. The problem of computing its dual basis $\{v_\lambda\}_{\lambda \vdash n}$ (i.e., $(u_\lambda, v_\mu) = \delta_{\lambda\mu}$) is equivalent to writing $\Omega_n(x;y)$ as a linear combination of u_λ's

- coefficients from $\mathbb{Z}[y_1, \dots, y_N]_n^{S_N}$ - are v_λ's.

$$\boxed{\begin{array}{l} \Omega_n(x;y) = \sum_{\lambda \vdash n} u_\lambda(x)\, v_\lambda(y) \\ \iff \{u_\lambda\} \ \&\ \{v_\lambda\} \text{ are 2 bases of } \Lambda_{n;\mathbb{Q}} \\ \text{dual to each other} \end{array}}$$