Lecture 9

Recall - Frobenius' characteristic map $F : (\mathbb{C}S_n)_{class} \xrightarrow{\sim} \Lambda_{n;\mathbb{C}}$ is an isometry

$$F(\delta_{c(\mu)}) = \frac{p_\mu}{z(\mu)} \quad (\forall \mu \vdash n)$$

$(\cdot,\cdot)$ on $(\mathbb{C}S_n)_{class}$: $\quad (f,g) = \sum_{\mu \vdash n} \frac{f(\mu)\, g(\mu)}{z(\mu)}$

$(\cdot,\cdot)$ on $\Lambda_{n;\mathbb{C}}$: $\quad (p_\lambda, p_\mu) = \delta_{\lambda\mu} \cdot z(\lambda)$

Canonical tensor of the inner product on $\Lambda_{n;\mathbb{C}}$ = degree n component of $\prod_{i,j} \frac{1}{1-x_i y_j}$.

(Here, $\Lambda_{n;\mathbb{C}} = \mathbb{C}[x_1,\dots,x_N]^{S_N}_{\text{degree } n}$; $\mu \vdash n$ means μ is a partition of n;

$z(\mu) = \prod j^{r_j} \cdot r_j!$ where $r_j = \#\{k \mid \mu_k = j\}$.)

§1. Statement of the main theorem .. Let $R(S_n) \subset (\mathbb{C}S_n)_{class}$ be defined as: $R(S_n) = \mathbb{Z}$-linear span of $\{\chi_V : V \in Rep_{fd}(S_n)\}$

Theorem. - (a) $F(R(S_n)) = \Lambda_n = \mathbb{Z}[x_1,\dots,x_N]^{S_N}_{deg=n}$.

(b) $\{s_\lambda\}_{\lambda \vdash n}$ is an orthonormal basis of $\Lambda_{n;\mathbb{C}}$.

(c) $s_\lambda \in \sum_{\mu \vdash n} \mathbb{Z}\, h_\mu$ $\quad$ [s_λ : Schur polynomials; h_μ : complete symmetric fn - see defn. below]

(d) $Ch(\chi_{U_\lambda}) = h_\lambda$ $\quad$ [U_λ : partition repn. of S_n - see Lecture 7, §2]

(e) Coefficient of $x_1 \cdots x_N$ in s_λ is positive.

Remarks and outline of the proof –

$$\begin{tikzcd}
(\mathbb{C}S_n)_{\text{class}} \arrow[r, "\mathcal{F}"] & \Lambda_{n;\mathbb{C}} \\
R(S_n) \arrow[u, phantom, "\cup"] \arrow[r, "\mathcal{F}", "(\text{Thm (a)})"'] & \Lambda_n \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Given $f \in (\mathbb{C}S_n)_{\text{class}}$, $f = \chi_V$ for some f.d. irred. repn. of S_n if and only if f satisfies three conditions – (i) $f \in R(S_n)$; (ii) $(f,f) = 1$
(iii) $f(e) > 0$.

- The theorem implies $\mathcal{F}^{-1}(s_\lambda) = \chi_{V_\lambda}$ for a unique f.d. irred. V_λ. and $\{V_\lambda\}_{\lambda \vdash n}$ is the complete set of all f.d. irred. repns. of S_n.

(Note, for $f \in (\mathbb{C}S_n)_{\text{class}}$, $\mathcal{F}(f) = \sum_{\mu \vdash n} f(\mu) \dfrac{p_\mu}{z(\mu)}$.

Hence, $f(e) = n!$ (Coefficient of $p_{(1,\dots,1)}$ in $\overset{\mathcal{F}}{\cancel{ch}}(f)$).

$p_{(1,\dots,1)} = (x_1 + \dots + x_N)^N$ is the only power sum symmetric function containing $x_1 \cdots x_N$. So Theorem (e) establishes criterion (iii) above).

- $\left.\begin{array}{l} s_\lambda \in \sum_{\mu \vdash n} \mathbb{Z}\, h_\mu \\ h_\mu = \mathcal{F}(1_\mu) \end{array}\right\}$ are needed to show that $s_\lambda \in \mathcal{F}(R(S_n))$.
Theorem (a) follows from the other (b–e) parts.; once we prove that $h_\mu \in \Lambda_n$ $\forall \mu \vdash n$.

§2. Schur polynomials and orthonormality.-

Definition. (Cauchy).- For $\underline{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, let

$$a_{\underline{\alpha}}(x_1, \dots, x_N) = \det\left(x_i^{\alpha_j}\right)_{1 \le i,j \le N}.$$

Let $\rho = (N-1, N-2, \dots, 0) \in \mathbb{N}^N$.

Note: (i) Each $a_{\underline{\alpha}}$ is a skew-symmetric polynomial of degree $\sum_{j=1}^{N} \alpha_j$.

i.e., $w \cdot a_{\underline{\alpha}} = \mathrm{sign}(w)\, a_{\underline{\alpha}} \quad \forall\, w \in S_N$.

Hence, if entries of $\underline{\alpha}$ are not distinct, then $a_{\underline{\alpha}} = 0$.

(ii) $\forall\, i \neq j$, $x_i - x_j$ divides $a_{\underline{\alpha}}$.

(iii) $a_\rho(x_1, \dots, x_N) = \prod_{i<j} (x_i - x_j)$ (Van der Monde determinant)

For every $\lambda \vdash n$, append 0's to make $\mathrm{length}(\lambda) = N$ and define

$$\boxed{s_\lambda(x_1, \dots, x_N) = \frac{a_{\lambda+\rho}(x_1, \dots, x_N)}{a_\rho(x_1, \dots, x_N)}}$$

Theorem. - The degree n component of $\prod_{i,j} \frac{1}{1 - x_i y_j}$ is equal to

$$\sum_{\lambda \vdash n} s_\lambda(x_1, \dots, x_N)\, s_\lambda(y_1, \dots, y_N)$$

[This implies (b) of Thm §1, according to the result of Lecture 8 page 7.]

<u>Proof.</u> – The proof is based on <u>Cauchy's determinant identity</u>:

$$\det\left(\frac{1}{1-x_i y_j}\right)_{1\le i,j\le N} = \frac{\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j)}{\prod_{i,j}(1-x_i y_j)}$$

(we leave the verification of Cauchy's identity as an exercise).

Thus, the statement of the theorem is equivalent to:

$$\text{Degree } n+|\rho| \text{ part of } \det\left(\frac{1}{1-x_i y_j}\right)_{1\le i,j\le N} = \sum_{\lambda\vdash n} a_{\lambda+\rho}(\underline{x})\, a_{\lambda+\rho}(\underline{y}). \quad -(*)$$

$\left(|\rho| = \frac{N(N-1)}{2}\right)$

Let us choose a $\lambda \vdash n$ and compute the coefficient of $x_1^{\lambda_1+N-1} x_2^{\lambda_2+N-2} \cdots x_N^{\lambda_N+0}$ in the left-hand side of $(*)$

(write $\alpha_1 = \lambda_1+N-1, \ldots, \alpha_N = \lambda_N+0$; so $\alpha_1 > \alpha_2 > \cdots > \alpha_N$)

L.H.S. of $(*)$: $\det\left(1 + x_i y_j + x_i^2 y_j^2 + \cdots\right)_{1\le i,j\le N}$

$$\text{Coeff. of } x_1^{\alpha_1} x_2^{\alpha_2}\cdots x_N^{\alpha_N} = \sum_{w\in S_N} (-1)^w\, y_{w(1)}^{\alpha_1}\cdots y_{w(N)}^{\alpha_N} \quad \text{(by defn. of det.)}$$

$$= a_{\lambda+\rho}(y) \quad \text{as claimed.} \qquad \square$$

§3. Complete and elementary symmetric fns.

For $r \in \mathbb{N}$, let $h_r(x_1, \dots, x_N) :=$ sum of all monomials of degree r.

$$= \sum_{\lambda \vdash r} m_\lambda .$$

and $e_r(x_1, \dots, x_N) := \sum_{1 \le i_1 < \dots < i_r \le N} x_{i_1} \cdots x_{i_r}$ (elementary symm. fns.)

Note: $h_0 = e_0 = 1$ and $e_\ell = 0 \ \forall\, \ell > N$. In terms of generating series:

$$H(t) = \sum_{r=0}^{\infty} h_r(x_1 \dots x_N)\, t^r = \prod_{i=1}^{N} (1 + x_i t + x_i^2 t^2 + \dots)$$

$$= \prod_{i=1}^{N} \frac{1}{1 - x_i t}$$

$$E(t) = \sum_{r=0}^{N} e_r(x_1, \dots, x_N)\, t^r = (1 + x_1 t) \cdots (1 + x_N t)$$

Theorem. (Jacobi–Trudi identity)

$$s_\lambda = \det\left(h_{\lambda_i - i + j} \right)_{1 \le i,j \le \ell} \qquad (\ell = \text{length of } \lambda).$$

[This proves Thm. §1 part (c)]

e.g. $$s_{(2,1)} = \det \begin{bmatrix} h_2 & h_3 \\ h_0 & h_1 \end{bmatrix} = h_2 h_1 - h_3$$

$$= \sum_{i,j} x_i^2 x_j + 2 \sum_{i<j<k} x_i x_j x_k$$

$$= m_{(2,1)} + 2\, m_{(1,1,1)} .$$

Note – positivity – not clear from defns. of s_λ.

Proof.- For $1 \leq k \leq N$, let $e_r^{(k)}$ denote the elementary symmetric polynomial in $N-1$ variables $x_1, \ldots, \underset{\substack{\uparrow \\ \text{skipped}}}{\hat{x}_k}, \ldots, x_N$. Thus,

$$\sum_{r=0}^{N-1} e_r^{(k)} t^r = \prod_{\substack{i=1 \\ (i \neq k)}}^{N} (1 + x_i t). \quad \text{From } H(t) = \sum_{r=0}^{\infty} h_r t^r = \prod_{i=1}^{N} \frac{1}{1-x_i t},$$

we get $\quad H(t) \cdot \sum_{r=0}^{N-1} (-1)^r e_r^{(k)} t^r = \dfrac{1}{1-x_k t} = \sum_{p=0}^{\infty} x_k^p t^p.$

<u>Coefficient of t^p</u>: $\quad \sum_{r=0}^{N-1} h_{p-r} (-1)^r e_r^{(k)} = x_k^p$

$\left[\text{convention: } h_{-\ell} = 0 \;\; \forall \ell \geq 1\right]$

Change r to $N-j$ to get

$$\sum_{j=1}^{N} h_{p-N+j} (-1)^{N-j} e_{N-j}^{(k)} = x_k^p.$$

Thus, for every $(\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ we get the following matrix equation.

$$\left(x_i^{\alpha_j}\right)_{1 \leq i,j \leq N} = \left((-1)^{N-r} e_{N-r}^{(i)}\right)_{1 \leq i,r \leq N} \left(h_{\alpha_j - N + r}\right)_{1 \leq r,j \leq N}$$

$$\Rightarrow \quad a_\alpha(x_1, \ldots, x_N) = \underset{\substack{\uparrow \\ \text{independent of } \alpha.}}{A} \cdot \det\left(h_{\alpha_j - N + r}\right)_{1 \leq r,j \leq N}$$

Set $\alpha = \rho = (N-1, \ldots, 0)$. Note $h_{\rho_j - N + r} = h_{r-j} = \begin{cases} 1 & \text{if } r = j \\ 0 & \text{if } r < j \end{cases}$

$$\Rightarrow \det\left(h_{\rho_j - N + r}\right) = 1.$$

So, we get $\quad a_\rho(x_1, \ldots, x_N) = A.$

Hence, $\quad a_\alpha(\underline{x}) = a_\rho(\underline{x}) \cdot \det\left(h_{\alpha_j - N + r}\right)_{1 \leq j, r \leq N}$

If $\alpha = \lambda + \rho$, then $\alpha_j - N + r = \lambda_j - j + r$, and we get

$$\frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})} = \det\left(h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq N}.$$

$$= \det\left(h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq l} \quad \text{since}$$

for $k > l$, $\lambda_k = 0$ and hence the remainder of the matrix after l rows is upper triangular with 1's on the diagonal

$$\left(h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq N} = \left[\begin{array}{c|ccc} \left(h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq l} & & & \\ \hline & 1 & & \\ \text{Zeroes} & 0 & \ddots & \\ & 0 & \cdots\ 0 & 1 \end{array}\right]$$

$\square$