Lecture 12

§1. Induced representations.-

Let G be a finite group and $H < G$ a subgroup. Note that any representation of G can be considered as a representation of H, via restriction. [Here, repns. are considered over $\mathbb{C}$].

$$\alpha : G \to GL(V) \rightsquigarrow \alpha\big|_H : H \to GL(V)$$

Thus, defining a functor $\mathrm{Res}_H^G : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(H)$.

Now, let $\rho : H \to GL(W)$ be a repn. of H

<u>Definition</u>.- <u>Induced representation</u> $V = \mathrm{Ind}_H^G(W)$ is defined as:

$$V = \{ f : G \to W \text{ s.t. } f(gh) = \rho(h)^{-1} \cdot f(g)\ \forall\, {}^{g\in G}_{h\in H} \}$$

(as vector space)

$G \circlearrowright V$ is given by:

$$(g\cdot f)(\sigma) = f(g^{-1}\sigma) \quad \forall\, f\in V;\ g,\sigma\in G.$$

[Check: $g\cdot f \in V$; i.e. $(g\cdot f)(\sigma h) = \rho(h)^{-1}\cdot((g\cdot f)(\sigma))$

$$(g\cdot f)(\sigma h) = f(g^{-1}\sigma h) = \rho(h)^{-1}\cdot f(g^{-1}\sigma) \checkmark \quad]$$

If $\varphi : W_1 \to W_2$ is an H-intertwiner, we get

$$\left.\begin{aligned} \mathrm{Ind}_H^G(\varphi) : \mathrm{Ind}_H^G(W_1) &\to \mathrm{Ind}_H^G(W_2) \\ f &\longmapsto \varphi\circ f \end{aligned}\right\} \text{ G-intertwiner.}$$

Ex: Check that Ind_H^G defined above gives a functor

$$\mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

§2. Examples and remarks.-

(i) Partition repns.- $G = S_n$ (symmetric group) ; $\lambda \vdash n$.

$$X_\lambda = \left\{ (I_1, \dots, I_\ell) \mid \{1,\dots,n\} = \coprod_{j=1}^{\ell} I_j \ ; \ |I_j| = \lambda_j \ \forall j \right\}$$

$$\longleftrightarrow \quad S_n / S_\lambda \quad \text{where } S_\lambda = S_{\lambda_1} \times \dots \times S_{\lambda_\ell} < S_n.$$

The partition repn U_λ from Lecture 10 is $\mathrm{Ind}_{S_\lambda}^{S_n}(\text{Trivial})$.

(ii) The process of taking induced repns. changes the underlying vector space. If $\dim W < \infty$, then,

$$\dim\left(\mathrm{Ind}_H^G W\right) = (\dim W) \cdot |G/H| .$$

(iii) Let $G = D_n$ (dihedral group $|D_n| = 2n$).

$$H = \langle r \rangle \cong \mathbb{Z}/_{n\mathbb{Z}} \quad < \quad G = D_n = \langle r, s \mid s^2 = r^n = (sr)^2 = e \rangle$$

Take $W = \mathbb{C}$ with $\rho_z : H \to GL(\mathbb{C}) = \mathbb{C}^\times$ $\quad (z \in \mathbb{C}^\times \text{ s.t. } z^n = 1)$

$$\rho_z(r) = z$$

$$V = \mathrm{Ind}_H^G(W) = \left\{ f : G \to \mathbb{C} \text{ s.t. } f(wr) = z^{-1} f(w) \ \forall w \in G \right\}$$

As $D_n = \left\{ \begin{matrix} e, r, r^2, \dots, r^{n-1} \\ s, sr, \dots, sr^{n-1} \end{matrix} \right\}$, V is 2-dim'l

$$V \simeq \mathbb{C} \oplus \mathbb{C} \qquad \left[\begin{array}{l} f(r^j) = z^{-j} f(e) \\ f(sr^j) = z^{-j} f(s) \end{array} \right].$$
$$f \longmapsto (f(e), f(s))$$

G-action on V is given by: (easy check)

$$s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad r \mapsto \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

§3. Frobenius' formula for characters of induced representations.

Theorem.- [G: finite group; $H < G$ a subgroup; $\rho : H \to GL(W)$ a f.d. repn]

$$\boxed{\chi_{\mathrm{Ind}_H^G(W)}(g) = \frac{1}{|H|} \sum_{\substack{x \in G: \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x)}$$

Proof.- Let $g_1, \ldots, g_\ell \in G$ be coset representatives in G/H, i.e. $G = \bigsqcup_{j=1}^{\ell} g_j H$. Then we have

$$V = \mathrm{Ind}_H^G(W) = \bigoplus_{j=1}^{\ell} W^{(j)} \quad \text{where}$$

$$W^{(j)} = \{ f \in V : f(g_i h) = 0 \ \forall i \neq j, h \in H \} \subset V.$$

Let $g \in G$ and $f \in W^{(j)}$. For $1 \leq k \leq l$ and $h \in H$,

we have: $$(g \cdot f)(g_k h) = f(g^{-1} g_k h) = 0 \text{ if } g^{-1} g_k \notin g_j H.$$

$$\Rightarrow g \cdot f \in W^{(k)} \text{ where } k \text{ is s.t. } g g_j H = g_k H.$$

Since trace only depends on diagonal blocks, we get:

$$\text{Trace of } g \text{ acting on } V = \sum_{j:\, g g_j H = g_j H} \text{Trace of } g \text{ acting on } W^{(j)}$$

$$= \sum_{j:\, g_j^{-1} g g_j \in H} \text{Trace of } g \text{ acting on } W^{(j)}.$$

Ex.: Verify that the following diagram commutes:

$$\begin{tikzcd} W^{(j)} \arrow[r, "f \mapsto f(g_j)", "\sim"'] \arrow[d, "g"'] & W \arrow[d, "g_j^{-1} g g_j"] \\ W^{(j)} \arrow[r, "\sim"] & W \end{tikzcd}$$

Hence,

$$\chi_{\mathrm{Ind}_H^G(W)}(g) = \sum_{\substack{j \text{ s.t.} \\ g_j^{-1} g g_j \in H}} \chi_W(g_j^{-1} g g_j)$$

Note: $\{ j : g_j^{-1} g g_j \in H \} \longleftrightarrow \{ x \in G : x^{-1} g x \in H \} / H$

(if $x^{-1} g x \in H$ then $(xh)^{-1} g (xh) \in H \ \forall h \in H$)

$$\Rightarrow \quad \chi_{\mathrm{Ind}_H^G(W)}(g) = \frac{1}{|H|} \sum_{\substack{x \in G: \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x)$$

$\square$

§4. Note that Frobenius' formula in fact defines a linear map

$$(\mathbb{C}H)_{class} \longrightarrow (\mathbb{C}G)_{class}$$

$$\psi \longmapsto \mathrm{Ind}_H^G(\psi) : g \mapsto \frac{1}{|H|} \sum_{\substack{x \in G: \\ x^{-1} g x \in H}} \psi(x^{-1} g x)$$

Our next result shows that induction is dual to restriction w.r.t. the usual inner product on class functions – i.e.,

$$(\phi, \mathrm{Ind}_H^G(\psi))_{\mathbb{C}G_{class}} = (\mathrm{Res}_H^G \phi, \psi)_{\mathbb{C}H_{class}} \quad \forall \begin{array}{l} \phi \in \mathbb{C}G_{class} \\ \psi \in \mathbb{C}H_{class} \end{array}$$

Theorem.- (Frobenius reciprocity):

There is a natural isomorphism of vector spaces; $\forall \begin{array}{l} V \in \mathrm{Rep}(G) \\ W \in \mathrm{Rep}(H) \end{array}$

$$\Phi : \mathrm{Hom}_H \left(\mathrm{Res}_H^G V \, , \, W \right) \longrightarrow \mathrm{Hom}_G \left(V , \mathrm{Ind}_H^G (W) \right)$$

given by :

$$\Phi(A)(v) \; : \; G \to W \qquad \forall \; A : V \to W$$
$$g \longmapsto A(g^{-1} \cdot v)$$

H-intertwiner, $v \in V$ and $g \in G$

Proof left as a routine verification.