Summary of results about repns. (over $\mathbb{C}$) of S_n :-

$$R(S_n) = \text{repn. ring of } S_n = \bigoplus_{\lambda\vdash n} \mathbb{Z}\,\chi_{V_\lambda} \quad \text{if } \{V_\lambda\}_{\lambda\vdash n} = \mathrm{Irred}_{fd}(S_n)$$

is isomorphic, as an abelian group, to $\Lambda_n = \mathbb{Z}[x_1,\dots,x_N]^{S_N}_{deg=n}$. $(N \geq n)$.

The identification $Ch : R(S_n) \to \Lambda_n$ is defined over $\mathbb{Q}$ as

$$Ch(\delta_{C(\lambda)}) = \frac{p_\lambda}{z(\lambda)} \quad \left(\text{recall: } z(\lambda) = |Z_{S_n}(w_\lambda)| \text{ for any } w_\lambda \in C_\lambda \right.$$

conj. class in S_n.

$$= \prod_{i\geq 1} i^{l_i}\cdot l_i!$$

$$\left.\text{where } l_i = \#\{j : \lambda_j = i\}\right)$$

Moreover, there is a unique irred. f.d. repn V_λ of S_n $(\forall \lambda\vdash n)$ s.t. $Ch(\chi_{V_\lambda}) = s_\lambda$ (Schur poly.)

Ch preserves $(\cdot,\cdot)$ non-deg. bilinear forms:

$(\chi_V, \chi_W) = \dim \mathrm{Hom}_{S_n}(V, W)$ on $R(S_n)$

$(p_\lambda, p_\mu) = \delta_{\lambda\mu}\, z(\lambda)$ on $\Lambda_{n;\mathbb{Q}}$ – restricted to Λ_n.

Various dual bases :
- $\{s_\lambda\}_{\lambda\vdash n}$ is an orthonormal basis of Λ_n.
- The bases dual to $\{m_\lambda\}_{\lambda\vdash n}$ is $\{h_\lambda\}_{\lambda\vdash n}$

monomial symm. fn.   complete symm. fn.

$$m_\lambda = \sum_{\alpha\in S_N\cdot\lambda} \underline{x}^\alpha \qquad h_r = \sum_{\substack{\alpha\in\mathbb{Z}^N_{\geq 0}\\ |\alpha|=r}} x^\alpha = \sum_{\gamma\vdash r} m_\gamma$$

$$h_\lambda = h_{\lambda_1}\cdot\dots\cdot h_{\lambda_\ell}$$

Characters of induced repn.

$$\lambda \vdash n \rightsquigarrow S_\lambda < S_n \rightsquigarrow U_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}(\text{Trivial}) = \mathrm{Fun}(S_n/S_\lambda ; \mathbb{C})$$

$i_\lambda = \chi_{U_\lambda} \in R(S_n)$. Then $ch(i_\lambda) = h_\lambda$.

Formulae for Schur polynomials. – Let $\rho = (N-1, N-2, \dots, 0) \in \mathbb{Z}_{\geq 0}^N$.
$\lambda = (\lambda_1, \dots, \lambda_\ell, 0 \dots, 0) \in \mathbb{Z}_{\geq 0}^N$.

$$s_\lambda(x_1, \dots, x_N) = \frac{a_{\lambda+\rho}(\underline{x})}{a_\rho(\underline{x})} = \frac{\det\left(x_i^{\lambda_j + N - j}\right)_{1 \leq i,j \leq N}}{\prod_{i<j} x_i - x_j}$$

$$= \det\left(h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq \ell}$$

$$= \sum_{\substack{t:\ \text{SSYT of} \\ \text{shape } \lambda, \text{ filled} \\ \text{with } \{1,\dots,N\}}} x^{v(t)} \qquad v(t)_i = \#\ i\text{'s in } t.$$

e.g. $\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \rightsquigarrow$ shape $(3,1,1)$, type $(2,1,0,1,1) \in \mathbb{Z}_{\geq 0}^5$

$$= \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu} = \#$ SSYT of shape λ and type μ.

$$h_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu \quad \text{where} \quad M_{\lambda\mu} = \left| \left\{ A = (a_{ij}) : \begin{array}{l} \sum_j a_{ij} = \lambda_i \\ \sum_i a_{ij} = \mu_j \end{array} \right\} \right|$$

§1. Pieri rule and its representation theoretic interpretation –

Let $\mu \vdash n-1$. Then $s_\mu s_1 = \sum_{\lambda \in U(\mu)} s_\lambda$. (Pieri rule left as an exercise)

$U(\mu)$ = set of partitions of n, obtained from μ – by adding a box.

We will now show that multiplication of symmetric polynomials, gives rise to – via ch – "induction product" on $\bigoplus_n R(S_n)$.

More precisely, let $m, n \in \mathbb{Z}_{\geq 0}$, then we have

$$R(S_m) \times R(S_n) \longrightarrow R(S_{m+n})$$
$$V_1, V_2 \longmapsto \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} (V_1 \otimes V_2) =: V_1 * V_2$$

§2. Theorem. – $\mathrm{ch}(\chi_{V_1 * V_2}) = \mathrm{ch}(\chi_{V_1}) \cdot \mathrm{ch}(\chi_{V_2})$.

(take number of variables $N \geq m+n$ for this proof).

Proof. – Using Frobenius' formula for characters of an induced repn.

$$\chi_{V_1 * V_2}(w) = \frac{1}{|S_m \times S_n|} \sum_{\substack{\pi \in S_{m+n} : \\ \pi^{-1} w \pi \in S_m \times S_n}} \chi_{V_1}(pr_1(\pi^{-1} w \pi)) \chi_{V_2}(pr_2(\pi^{-1} w \pi))$$

$$S_m \times S_n \xrightarrow{pr_1} S_m, \quad S_m \times S_n \xrightarrow{pr_2} S_n$$

Now, $\{\pi \in S_{m+n} : \pi^{-1} w \pi \in S_m \times S_n\} \xrightarrow{f}$ Conj. class of w $\cap$ $S_m \times S_n$

has fibers = Centralizer of w in S_{m+n}

Therefore, $\chi_{V_1 * V_2}(w) = \sum_{\substack{\lambda \vdash m \\ \mu \vdash n \text{ s.t. } \lambda \cup \mu = \text{cycle type of } w \text{ (say } \gamma)}} \chi_{V_1}(\lambda)\chi_{V_2}(\mu) \cdot \frac{|C(\lambda)|}{m!} \frac{|C(\mu)|}{n!} z(\gamma)$

By defn of Ch $\left(\mathrm{Ch}(f) = \sum_{\lambda \vdash n} f(\lambda) \frac{p_\lambda}{z(\lambda)} \right)$:

$$\mathrm{Ch}(\chi_{V_1 * V_2}) = \sum_{\gamma \vdash m+n} \chi_{V_1 * V_2}(\gamma) \cdot \frac{p_\gamma}{z(\gamma)}$$

$$= \sum_{\gamma \vdash m+n} \left(\sum_{\substack{\lambda \vdash m \\ \mu \vdash n \text{ s.t.} \\ \gamma = \lambda \cup \mu}} \frac{\chi_{V_1}(\lambda)}{z(\lambda)} \frac{\chi_{V_2}(\mu)}{z(\mu)} \right) p_\gamma$$

$$= \left(\sum_{\lambda \vdash m} \chi_{V_1}(\lambda) \frac{p_\lambda}{z(\lambda)} \right) \left(\sum_{\mu \vdash n} \frac{\chi_{V_2}(\mu) p_\mu}{z(\mu)} \right) \quad [\, p_\gamma = p_\lambda \cdot p_\mu \,].$$

$\square$

§3. Cor. - (a) For $\mu \vdash (n-1)$, $\mathrm{Ind}_{S_{n-1}}^{S_n}(V_\mu) = \bigoplus_{\lambda \in U(\mu)} V_\lambda$

[Branching rules]

(b) For $\lambda \vdash n$; $\mathrm{Res}_{S_{n-1}}^{S_n}(V_\lambda) = \bigoplus_{\mu \in D(\lambda)} V_\mu$

($D(\lambda)$ = partitions of $n-1$, obtained from λ by removing a box).

Pf. - (a) Apply Ch on both sides and use Thm §2., combined with Pieri rule from §1.

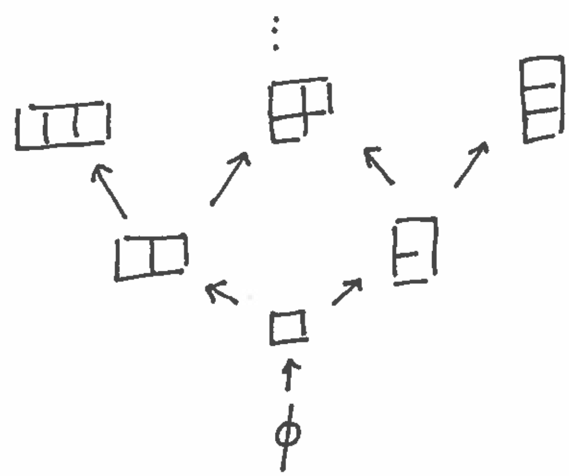
(b) For $\mu \vdash (n-1)$, coefficient (or multiplicity) of V_μ in $\mathrm{Res}^{S_n}_{S_{n-1}}(V_\lambda)$ = Coefficient of V_λ in $\mathrm{Ind}^{S_n}_{S_{n-1}}(V_\mu)$

by Frobenius reciprocity

$= 1 \iff \lambda \in U(\mu) \iff \mu \in D(\lambda)$. $\square$

§4. Young lattice.– Y : partially ordered set of all partitions. $Y = \bigsqcup_{n=0}^{\infty} \mathrm{Part}(n)$

Strict, minimal rel$^{\underline{n}}$ $\lambda \leftarrow \mu$ means $\lambda \in U(\mu)$



(Young lattice)

Branching rules

$$\Rightarrow \dim V_\lambda = \sum_{\substack{\mu \vdash n-1 \\ \lambda \in U(\mu)}} \dim V_\mu$$
$(\lambda \vdash n)$

$= \cdots =$ # of paths $\phi \to \lambda$ in the Young lattice.

$=$ # SYT of shape λ (by remembering where i-th box was added).

<u>Ex.</u>: Let $\mathcal{F}$ be a $\mathbb{Q}$-vector space spanned by $\{|\lambda\rangle : \lambda \in Y\}$.

Consider $U, D \in \mathrm{End}(\mathcal{F})$ given by
$$U(\mu) = \sum_{\lambda \in U(\mu)} |\lambda\rangle$$
$$D(\lambda) = \sum_{\mu \in \mathcal{D}(\lambda)} |\mu\rangle$$

Show that $DU - UD = \mathrm{Id}$.

§5. References and suggested further readings.

- Etingof et al - Introduction to Representation Theory.
  ~~Ch 4~~ § 5.12, 5.13 - Young symmetrizer approach towards Repn. Th. of Sn.
- A. Prasad - Representation theory: a combinatorial viewpoint
- R. Stanley - Differential posets (Journal of the American Math. Soc. 1988)
- I. G. Macdonald - Symmetric functions & Hall polynomials.
- A. Zelevinsky - Repns. of finite classical groups - a Hopf alg. approach (Lecture Notes in Math. 869)
- A. M. Vershik & A. Yu. Okounkov - A new approach to the repn. th. of symmetric groups (ArXiv: 0503040 ; Selecta Math. (1996))