§1. Polynomial functors. Let Vect be the category of finite-dimensional $\mathbb{C}$-vector spaces. (All arguments given here are valid for any field of characteristic zero).

A polynomial functor $F : \underline{Vect} \to \underline{Vect}$ is a functor – i.e.,

(definition of a covariant functor)
$$\left[\begin{array}{l} \forall\ X \in \underline{Vect} \text{ we have } F(X) \in \underline{Vect} \\ \forall\ f \in \mathrm{Hom}_{\mathbb{C}}(X,Y) \text{ we have } F(f) \in \mathrm{Hom}_{\mathbb{C}}(F(X),F(Y)) \text{ s.t.} \\ F(Id_X) = Id_{F(X)} \text{ and } F(g\circ f) = F(g)\circ F(f) \end{array}\right.$$

s.t. $\forall\ V, W \in \underline{Vect}$, the map ~~F(f)~~ $f \mapsto F(f)$ is a polynomial map.

i.e., $F : \mathrm{Hom}_{\mathbb{C}}(V,W) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(F(V),F(W))$

$\forall\ f : V \to W$, the entries of $F(f) : F(V) \to F(W)$ are polynomials in the entries of f (viewed as matrices - after choosing bases of relevant vector spaces)

Alternately, F is a polynomial functor if $\forall\ f_1, \dots, f_r : V \to W$ in Vect fixed, and $\lambda_1, \dots, \lambda_r \in \mathbb{C}$,

$$(\lambda_1, \dots, \lambda_r) \longmapsto F(\lambda_1 f_1 + \dots + \lambda_r f_r) \text{ is a polynomial}$$

in $\lambda_1, \dots, \lambda_r$ (entries from $\mathrm{Hom}_{\mathbb{C}}(F(V),F(W))$).

We say F is homogeneous of degree n if $F(\lambda_1 f_1 + \dots + \lambda_r f_r)$ is homogeneous poly. in $\lambda_1, \dots, \lambda_r$ of degree n ; $\forall\ r \geq 1$, $\forall f_1, \dots, f_r : V \to W$.

§2. Example. - <u>tensor product</u>. For $n \in \mathbb{Z}_{\geq 0}$, let

$T^n : \underline{Vect} \to \underline{Vect}$ be given by

- $T^n(V) = \underbrace{V \otimes \cdots \otimes V}_{n\text{-times}}$ (Convention: $T^0(V) = \mathbb{C} \ \forall V \in \underline{Vect}$, $T^0(f) = Id_{\mathbb{C}} = 1 \ \forall f : V \to W$)

If $\{v_1, \ldots, v_m\}$ is a basis of V, then

$$\left\{ v_{i_1 \ldots i_n} = v_{i_1} \otimes \cdots \otimes v_{i_n} \ : \ i_1, \ldots, i_n \in \{1, \ldots, m\} \right\}$$

is a basis of $T^n(V)$.

- For $f : V \to W$ ($\mathbb{C}$-linear), $T^n(f) : \underbrace{V \otimes \cdots \otimes V}_{n\text{-times}} \to \underbrace{W \otimes \cdots \otimes W}_{n\text{-times}}$ is

given by $T^n(f) : a_1 \otimes \cdots \otimes a_n \longmapsto f(a_1) \otimes \cdots \otimes f(a_n)$

Let $\{v_1, \ldots, v_m\}$ be a basis of V
$\{w_1, \ldots, w_\ell\}$ be a basis of W.

$f : V \to W$ is given by $\ell \times m$ matrix $(f_{ij})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}}$:

$$f v_j = \sum_{i=1}^{\ell} f_{ij} w_i \quad \forall \ 1 \leq j \leq m.$$

Then
$$(T^n f)(v_{j_1 \cdots j_n}) = \sum_{\substack{\underline{i} = (i_1, \ldots, i_n) \\ \in \{1, \ldots, \ell\}^n}} w_{i_1 \ldots i_n} \cdot \underbrace{\left(f_{i_1 j_1} \cdots f_{i_n j_n}\right)}_{\substack{\text{hgs. degree } n \\ \text{polynomial in the} \\ \text{entries of } f.}}$$

§3. Motivation and remarks. -

Given a polynomial functor $F : \underline{Vect} \to \underline{Vect}$, we get a <u>polynomial representation</u> of $GL_m(\mathbb{C})$ ($\forall$ m)

$$F \longmapsto F(\mathbb{C}^m) = L$$

$GL_m(\mathbb{C})$ acts on L via $g \in GL_m(\mathbb{C}) \subset \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^m, \mathbb{C}^m)$
$\longmapsto F(g) : L \to L$

(A <u>polynomial repn.</u> of $GL_m(\mathbb{C})$ is a f.d. vector space L, together with a group hom. $\alpha : GL_m(\mathbb{C}) \to GL(L) = \mathrm{Aut}_{\mathbb{C}\text{-v.s.}}(L)$ s.t. $\forall\, g \in GL_m(\mathbb{C})$, matrix entries of $\alpha(g)$ are polynomials in the entries of g.)

§4. <u>Lemma.</u> (Schur) -

Let $F : \underline{Vect} \to \underline{Vect}$ be a polynomial functor. Then there exist hgs poly. functors $\{ \underbrace{F_n : \underline{Vect} \to \underline{Vect}}_{\text{hgs. of deg. } n} \}_{n \geq 0}$ and natural iso.

$$F \simeq F_0 \oplus F_1 \oplus \cdots$$

(i.e. $\forall\, V \in \underline{Vect}$ we have an iso. $\alpha_V : F(V) \to \bigoplus_{n \geq 0} F_n(V)$ s.t. $\forall\, f : V \to W$,

$$\begin{array}{ccc} F(V) & \xrightarrow{\alpha_V} & \bigoplus_{n\geq 0} F_n(V) \\ F(f) \downarrow & & \downarrow \oplus F_n(f) \\ F(W) & \xrightarrow[\alpha_W]{} & \bigoplus_{n\geq 0} F_n(W) \end{array}$$

commutes)

Proof. – For $X \in \underline{\text{Vect}}$ and $\lambda \in \mathbb{C}$, let $\lambda_X = \lambda \cdot Id_X \in Hom_{\mathbb{C}}(X,X)$.

By defn. $F(\lambda_X)$ is a polynomial in λ with coefficents from $End_{\mathbb{C}}(F(X))$:

$$F(\lambda_X) = \sum_{n\geq 0} u_n(X)\cdot \lambda^n \quad ; \quad u_n(X) : F(X) \to F(X)$$

As F is a functor, $F(Id_X) = Id_{F(X)} \Rightarrow \sum_{n\geq 0} u_n(X) = Id_{F(X)}$

$$F((\lambda\mu)_X) = F(\lambda_X)\, F(\mu_X)$$

$$\Rightarrow \sum_{n\geq 0} u_n(X)\, \lambda^n \mu^n = \Big(\sum_{k\geq 0} u_k(X)\, \lambda^k\Big)\Big(\sum_{\ell\geq 0} u_\ell(X) \mu^\ell\Big)$$

Comparing coefficients, we get $u_k(X)\, u_\ell(X) = 0$ if $k \neq \ell$
$u_n(X)^2 = u_n(X)$ $\forall\, n \geq 0$.

Hence $F(X) = \bigoplus_{n\geq 0} \text{Image}(u_n(X))$.

Define: $F_n(X) := \text{Image of } u_n(X)$.

$\forall f : X \to Y$ in $\underline{\text{Vect}}$, $F(\lambda_Y \circ f) = F(f \circ \lambda_X)$

$\Rightarrow f$ commutes with u_n

(i.e., $f \circ u_n(X) = u_n(Y) \circ f$)

So, $f|_{F_n(X)} : F_n(X) \to F_n(Y)$. (Check: F_n is a functor).

$\underline{\text{Easy exercise}}$: $F \simeq \bigoplus_{n\geq 0} F_n$ $\square$

§5. Linearization.-

Let F be a hgs. poly. functor of degree $n \geq 1$.

$$F : \underline{\mathrm{Vect}} \to \underline{\mathrm{Vect}}.$$

Consider
$$\widetilde{F} : \underline{\mathrm{Vect}}^n \to \underline{\mathrm{Vect}}$$
$$(X_1, \dots, X_n) \mapsto F(X_1 \oplus \dots \oplus X_n)$$

[Note: $\underline{\mathrm{Vect}}^n$ is a category whose objects are n-tuples of f.d. vector spaces, and morphisms are n-tuples of linear maps.

$$\underline{X} = (X_1, \dots, X_n), \quad \underline{Y} = (Y_1, \dots, Y_n) \rightsquigarrow \mathrm{Hom}(\underline{X}, \underline{Y}) = \prod_{i=1}^{n} \mathrm{Hom}_{\mathbb{C}}(X_i, Y_i).\ ]$$

For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, $(X_1, \dots, X_n \in \underline{\mathrm{Vect}})$, $F(\lambda_1 \mathrm{Id}_{X_1} \oplus \dots \oplus \lambda_n \mathrm{Id}_{X_n})$ has total degree n in $\lambda_1, \dots, \lambda_n$

$$= \sum_{\substack{m_1, \dots, m_n \in \mathbb{N}^n : \\ m_1 + \dots + m_n = n}} u_{m_1, \dots, m_n}(X_1, \dots, X_n) \cdot \lambda_1^{m_1} \cdots \lambda_n^{m_n}$$

as in the proof of Lemma §4 above.

Let $L_F : \underline{\mathrm{Vect}}^n \to \underline{\mathrm{Vect}}$ denote the $m_1 = \dots = m_n = 1$ component of the decomposition $\widetilde{F} = \bigoplus_{m_1, \dots, m_n} \widetilde{F}_{m_1, \dots, m_n}$. [called *linearization* of F.]

Let $v = u_{1, \dots, 1}$ i.e. $\forall X_1, \dots, X_n \in \underline{\mathrm{Vect}}$

$v(X_1, \dots, X_n)$ is the coeff. of $\lambda_1 \cdots \lambda_n$ in $F\left(\bigoplus_{j=1}^{n} \lambda_j \mathrm{Id}_{X_j}\right)$

$$\in \mathrm{End}_{\mathbb{C}}\left(F(X_1 \oplus \dots \oplus X_n)\right)$$

In more detail, let $X_1, \dots, X_n \in \underline{\mathrm{Vect}}$; $Y = X_1 \oplus \dots \oplus X_n$

$i_\alpha : X_\alpha \to Y$; $p_\alpha : Y \to X_\alpha$ natural inclusions & projections

(Note. $p_\alpha i_\alpha = Id_{X_\alpha}$; $p_\alpha i_\beta = 0$ if $\alpha \neq \beta$; $\sum i_\alpha p_\alpha = Id_Y$.)

For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, let $(\lambda)_Y : Y \to Y$ be given by $\sum_{j=1}^{n} \lambda_j Id_{X_j}$. $\left(= \sum_\alpha \lambda_\alpha p_\alpha i_\alpha\right)$

$v(X_1, \dots, X_n) =$ coeff. of $\lambda_1 \cdots \lambda_n$ in $F((\lambda)_Y)$.

$L_F(X_1, \dots, X_n) =$ Image of $v(X_1, \dots, X_n)$.

e.g. $F(V) = Sym^2(V) = $ Span of $\{v \otimes v : v \in V\} \subset V \otimes V$

(basis given by $v_i \cdot v_j : 1 \leq i \leq j \leq \dim V$)

hgs. of degree 2 where $v_i \cdot v_j = \dfrac{v_i \otimes v_j + v_j \otimes v_i}{2} = \dfrac{(v_i + v_j) \otimes (v_i + v_j) - v_i \otimes v_i - v_j \otimes v_j}{2}$

$\{v_1, \dots, v_N\}$ a basis of V.

$$F(V \oplus W) = Sym^2(V \oplus W)$$
$$\cong Sym^2(V) \oplus (V \otimes W) \oplus Sym^2(W)$$

Note $Sym^2(\lambda Id_V + \mu Id_W)$:

		deg.
$v_i v_j \longmapsto \lambda^2 v_i v_j$		$(2,0)$
$v_i w_j \longmapsto \lambda\mu\, v_i w_j$		$(1,1)$
$w_j w_k \longmapsto \mu^2 w_j w_k$		$(0,2)$

$L_F =$ Projection onto $V \otimes W$ summand

$L_F(V, W) = V \otimes W$.