§1. Linearization of homogeneous polynomial functors.-

Recall: $\underline{Vect}$ = category of finite-dim'l $\mathbb{C}$-vector spaces.

A functor $F : \underline{Vect} \to \underline{Vect}$ is a polynomial functor if $\forall\, X, Y \in \underline{Vect}$, $r \geq 1$, and $f_1, \dots, f_r : X \to Y$ linear maps,

$F(\lambda_1 f_1 + \dots + \lambda_r f_r)$ is a polynomial in $\lambda_1, \dots, \lambda_r$ (with coefficients from $\mathrm{Hom}_{\mathbb{C}}(F(X), F(Y))$.)

F is said to be homogeneous of degree $n \in \mathbb{N}$, if $F(\lambda_1 f_1 + \dots + \lambda_r f_r)$ is hgs. of deg. n in $\lambda_1, \dots, \lambda_r$.

By Lemma §4 of Lecture 14, every poly functor is a direct sum of hgs poly. functors.

Let $n \in \mathbb{Z}_{\geq 1}$ and Poly_n = category of hgs. poly functors of degree n.

[Objects: $F : \underline{Vect} \to \underline{Vect}$ hgs. poly. deg $= n$.

Morphisms: $\xi : F_1 \to F_2$ natural transformations

- i.e. $\xi_V \in \mathrm{Hom}(F_1(V), F_2(V))$ $\forall\, V \in \underline{Vect}$

s.t.
$$\begin{tikzcd} F_1(V) \arrow[r, "\xi_V"] \arrow[d, "F_1(f)"'] & F_2(V) \arrow[d, "F_2(f)"] \\ F_1(W) \arrow[r, "\xi_W"'] & F_2(W) \end{tikzcd}$$
commutes $\forall\, f : V \to W$]

$F \in \mathrm{Poly}_n \rightsquigarrow L_F : \underline{Vect}^n \to \underline{Vect}$ linear in each argument

(linearization of F)

$L_F(X_1, \dots, X_n)$ was defined as follows. Consider the decomposition:

$$F(\lambda_1 Id_{X_1} \oplus \dots \oplus \lambda_n Id_{X_n}) = \sum_{\substack{m_1, \dots, m_n \in \mathbb{N} \\ m_1 + \dots + m_n = n}} \lambda_1^{m_1} \cdots \lambda_n^{m_n} \underbrace{\left(u_{\underline{m}}(\underline{X}) \right)}_{\in End(F(X_1 \oplus \dots \oplus X_n))}$$

As in the proof of Lemma §4, Lecture 14, complete set of pairwise orthogonal idempotents

$$F(X_1 \oplus \dots \oplus X_n) \simeq \bigoplus_{\substack{m_1, \dots, m_n \\ \text{as above}}} F_{m_1, \dots, m_n}(X_1, \dots, X_n)$$

naturally in $X_1, \dots, X_n$. $\quad L_F := F_{1, \dots, 1}$

<u>Notation</u> $\quad v(\underline{X}) = u_{1, \dots, 1}(\underline{X}) : F(\oplus X_j) \to F(\oplus X_j)$

is the projection onto $L_F(X_1, \dots, X_n)$ summand.

§2. <u>Action of the symmetric group</u> – For every $X \in \underline{Vect}$, the vector space $L_F^{(n)}(X) := L_F(X, \dots, X)$ carries a natural (in X) action of the symmetric group S_n. To describe it, let us introduce some notations.

$$Y = \underbrace{X \oplus \dots \oplus X}_{n \text{ copies}} \qquad \forall\, 1 \le \alpha \le n, \quad \begin{array}{l} i_\alpha : X \to Y \\ p_\alpha : Y \to X \end{array} \quad \text{canonical inclusion \& projection}$$

Note: $\quad p_\beta \circ i_\alpha = \delta_{\alpha\beta} \cdot Id_X \qquad \sum_\alpha i_\alpha p_\alpha = Id_Y$

For $w \in S_n$, let $\sigma_w : Y \to Y$ be given by (recall $Y = X^{\oplus n}$)

$$\sigma_w = \sum_{\alpha=1}^{n} i_{w(\alpha)} \circ p_\alpha .$$

[σ_w simply permutes the direct summands of $X^{\oplus n}$.]

Therefore, we get $F(\sigma_w) \in F(Y)$.

Note: for $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, if $D_\lambda = \sum_{\alpha=1}^{n} \lambda_\alpha Id_X = \sum_{\alpha=1}^{n} \lambda_\alpha i_\alpha p_\alpha \in End(Y)$

then $$\boxed{\sigma_w \circ D_\lambda = D_{w\lambda} \circ \sigma_w} \qquad w\lambda = (\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)}).$$

(Proof.-

$$\sigma_w \circ D_\lambda = \sum_{\alpha,\beta} i_{w(\alpha)} \, p_\alpha \, \lambda_\beta \, i_\beta \, p_\beta \qquad (p_\alpha i_\beta = \delta_{\alpha\beta} Id)$$

$$= \sum_\alpha \lambda_\alpha \, i_{w(\alpha)} \, p_\alpha$$

$$D_{w\lambda} \circ \sigma_w = \sum_{\alpha,\beta} \lambda_{w^{-1}(\alpha)} \, i_\alpha \underbrace{p_\alpha \, i_{w(\beta)}}_{= \begin{cases} Id & \text{for } \alpha = w\beta \\ 0 & \text{o/w} \end{cases}} p_\beta$$

$$= \sum_\beta \lambda_\beta \, i_{w(\beta)} \, p_\beta . \quad)$$

Applying F gives $$\boxed{F(\sigma_w) \circ F(D_\lambda) = F(D_{w\lambda}) \circ F(\sigma_w)}$$

Taking coefficient of $\lambda_1 \cdots \lambda_n$ on both sides implies that $F(\sigma_w)$ preserves the summand $L_F^{(n)}(X)$ of $F(Y)$.

i.e. $\underline{F(\sigma_w) \text{ commutes with } \vartheta}$.

Let $F'(\sigma_w)$ denote this restriction. That is, if $j_X : L_F^{(n)}(X) \hookrightarrow F(Y)$, $q_X : F(Y) \twoheadrightarrow L_F^{(n)}(X)$ are natural inclusions & projection

then, $$\boxed{F'(\sigma_w) = q_X \circ F(\sigma_w) \circ j_X}$$

$$\left(\begin{array}{l} q \circ j = Id \text{ on } L_F^{(n)}(X) \\ j \circ q = \vartheta \end{array} \right)$$

In conclusion, $w \longmapsto F'(\sigma_w)$ gives a natural S_n-action on $L_F^{(n)}(X)$. (Easy exercise · check $F'(\sigma_{w_1 w_2}) = F'(\sigma_{w_1}) F'(\sigma_{w_2})$.)

§3. Recovering F from $L_F^{(n)}$. – ($X \in \underline{\text{Vect}}$; $Y = X^{\oplus n}$)
$i_\alpha : X \to Y \quad p_\alpha : Y \to X$ as before

Let $i : X \to Y$ and $p : Y \to X$ be defined by $i = \sum_\alpha i_\alpha$; $p = \sum_\alpha p_\alpha$. Consider the natural transformations

$$\xi = q \circ F(i) : F \to L_F^{(n)}$$
$$\eta = F(p) \circ j : L_F^{(n)} \to F$$

$\left(\text{recall: } L_F^{(n)}(X) \underset{q}{\overset{j}{\rightleftarrows}} F(Y)\right)$

Theorem. – $\eta \circ \xi = $ mult. by $n!$

$$\xi \circ \eta = \sum_{w \in S_n} F'(\sigma_w)$$ as defined in §2 above.

Proof. – $\eta \circ \xi = F(p) \circ j \circ q \circ F(i) = F(p) \circ v \circ F(i)$.

Thus, for a given $X \in \underline{\text{Vect}}$ ($Y = X^{\oplus n}$), $(\eta \circ \xi)_X$ is the coefficient of $\lambda_1 \cdots \lambda_n$ in $F(p) \circ F(D_\lambda) \circ F(i)$

$$= F(p \circ D_\lambda \circ i)$$

$[D_\lambda = \sum_\alpha \lambda_\alpha i_\alpha p_\alpha]$

Now $p \circ D_\lambda \circ i = $ scalar mult. by $\lambda_1 + \dots + \lambda_n$

$$\Rightarrow F(p \circ D_\lambda \circ i) = (\lambda_1 + \dots + \lambda_n)^n \cdot \mathrm{Id}_{F(X)}$$

Coeff. of $\lambda_1 \cdots \lambda_n$ in $(\lambda_1 + \dots + \lambda_n)^n = n!$.

For the converse, $\xi \circ \eta = q \circ F(i \circ p) \circ j$

Claim. - $v \circ F(i \circ p) \circ v = \sum_{w \in S_n} F(\sigma_w) \circ v$

Given the claim, we have: $j \circ (\xi \circ \eta) \circ q = v \circ F(i \circ p) \circ v$ $(v = j \circ q)$

$$= \sum_{w \in S_n} F(\sigma_w) \circ v$$

As $q \circ j = Id$, we get $\xi \circ \eta = \sum_{w \in S_n} q \circ F(\sigma_w) \circ j = \sum_{w \in S_n} F'(\sigma_w)$.

Proof of the claim. - Consider linear transformations of the form $\sum_{\alpha, \beta} f_{\alpha\beta} \, i_\alpha p_\beta : Y \to Y$. Thus $F(\sum f_{\alpha\beta} i_\alpha p_\beta)$ will be a polynomial ~~is~~ of degree n in n^2 variables $\{f_{\alpha\beta}\}$. For $w \in S_n$, let $C_w \in End(Y)$ be the coefficient of $f_{w(1),1} \cdots f_{w(n),n}$ in $F(\sum f_{\alpha\beta} i_\alpha p_\beta)$.

Now $F(\sigma_w) \circ v = v \circ F(\sigma_w) \circ v$ [as $v^2 = v$ and $F(\sigma_w)$ commutes with v.]

is, by definition, the coefficient of $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$ in

$$F(D_\lambda) F(\sigma_w) F(D_\mu) = F(D_\lambda \circ \sigma_w \circ D_\mu)$$

$$= \sum_\alpha \lambda_{w(\alpha)} \mu_\alpha \, i_{w(\alpha)} p_\alpha$$

Hence, $F(\sigma_w) \circ v = C_w$ as above.

On the other hand, $v \circ F(i \circ p) \circ v$ is the coefficient of $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$

in $F(D_\lambda\, i\, p\, D_\mu) = F\Big(\sum_{\alpha,\beta} \lambda_\alpha \mu_\beta\, i_\alpha\, p_\beta\Big)$

which is $\sum_{w\in S_n} c_w$. The claim is proved. $\square$

§4. Corollary.- We have functorial isomorphisms:

$$F(X) \simeq L_F^{(n)}(X)^{S_n}$$

(The averaging operator $\frac{1}{n!}\sum_{w\in S_n} w = (n!)^{-1}\, \xi \circ \eta$

is idempotent, with image $= L_F^{(n)}(X)^{S_n}$)

Hence $F \in \mathrm{Poly}_n$ is of the form $X \mapsto L(X,\dots,X)^{S_n}$ where

$L : \underline{\mathrm{Vect}}^n \to \underline{\mathrm{Vect}}$ is hgs of degree 1 in each variable.

§5. Theorem. Let $L : \underline{\mathrm{Vect}}^n \to \underline{\mathrm{Vect}}$ be n-multi-linear. Then we have functorial iso.

$$L(X_1,\dots,X_n) \cong \overbrace{L(\mathbb{C},\dots,\mathbb{C})}^{L^{(n)}(\mathbb{C})} \otimes X_1 \otimes \dots \otimes X_n .$$

Proof. - ($n=1$ case): $L : \underline{\mathrm{Vect}} \to \underline{\mathrm{Vect}}$ is linear, hence

$L(f_1+f_2) = L(f_1) + L(f_2) \quad \forall\ f_1, f_2 : X_1 \to X_2$ in $\underline{\mathrm{Vect}}$.

For $x \in X$, let $e_x : \mathbb{C} \to X$, $\lambda \mapsto \lambda x$. Consider the natural map

$$\psi_X : \mathrm{Hom}_{\mathbb{C}}(L(X), Y) \to \mathrm{Hom}_{\mathbb{C}}(X, \mathrm{Hom}_{\mathbb{C}}(L(\mathbb{C}), Y))$$

$$(\psi_X f)(x) = f \circ L(e_x).$$

Combining with tensor-hom adjointness, we have

$$\phi_X : \mathrm{Hom}_{\mathbb{C}}(L(X), Y) \to \mathrm{Hom}_{\mathbb{C}}(L(\mathbb{C}) \otimes X, Y)$$

We claim that ϕ_X is an isomorphism. Since L is additive, if ϕ_{X_1} and ϕ_{X_2} are iso.'s then so is $\phi_{X_1 \oplus X_2}$. Hence it is enough to show that $\phi_{\mathbb{C}}$ is an iso, which is obvious.

Now we can inductively argue $(n \geq 1)$:

$$L(X_1, \ldots, X_n) \cong L(X_1, \ldots, X_{n-1}, \mathbb{C}) \otimes X_n$$

$$\ldots \cong L(\mathbb{C}, \ldots, \mathbb{C}) \otimes X_1 \otimes \ldots \otimes X_n.$$

□

Conclusion.- $\forall\ F \in \mathrm{Poly}_n$, we have functorial iso.'s

$$F(X) \cong \left(L_F^{(n)}(\mathbb{C}) \otimes T^n(X)\right)^{S_n}.$$