Lecture 16

Recall that we proved the following results about polynomial functors:

(1) Every poly. functor splits naturally as a direct sum of homogeneous poly. functors.

Let Poly_n = category of hgs. poly. functors of degree n.

$\forall F \in \mathrm{Poly}_n$, we defined $L_F : \mathrm{Vect}^n \to \mathrm{Vect}$ as the $(1, \dots, 1)$ component of $F(X_1 \oplus \dots \oplus X_n)$. $L_F^{(n)}(X) := L_F(X, \dots, X)$.

(2) We have natural transformations

$$\xi = q \circ F(i) : F \to L_F^{(n)}$$

$$\eta = F(p) \circ j : L_F^{(n)} \to F$$

[here $L_F^{(n)}(X) \underset{q_X}{\overset{j_X}{\rightleftarrows}} F(X^{\oplus n})$

$i_X : X \to X^{\oplus n}$ $\quad p_X : X^{\oplus n} \to X$

$i_X(x) = (x, \dots, x)\ \forall x \in X$

$p_X(x_1, \dots, x_n) = \sum_{j=1}^{n} x_j$]

(Thm §3 Lecture 15)

$$\eta \circ \xi = \text{mult. by } n!$$

$$\xi \circ \eta = \sum_{w \in S_n} F'(\sigma_w)$$

$F'(\sigma_w) \circlearrowright L_F^{(n)}$ natural S_n-action.

(3) (Cor §4 Lecture 15) We have natural iso.

$$F(X) \cong L_F^{(n)}(X)^{S_n}$$

(4) Every n-linear $L : \underline{\mathrm{Vect}}^n \to \underline{\mathrm{Vect}}$ is of the form (Thm §5 L15)

$$L(X_1, \dots, X_n) = V \otimes X_1 \otimes \dots \otimes X_n$$

where $V = L(\mathbb{C}, \dots, \mathbb{C})$ (if $L = L_F$, $V = L_F^{(n)}(\mathbb{C})$ is an S_n-repn.)

§1. <u>Theorem.</u> The following are inverse functors to each other:

$$\varphi : \mathrm{Poly}_n \longrightarrow S_n\text{-repns} \qquad \psi : S_n\text{-repns} \longrightarrow \mathrm{Poly}_n$$
$$F \longmapsto L_F^{(n)}(\mathbb{C}) \qquad\qquad M \mapsto \psi(M) : X \mapsto \left(M \otimes X^{\otimes n}\right)^{S_n}.$$

<u>Proof.</u> - $(\psi \circ \varphi)(F) : X \longmapsto \left(L_F^{(n)}(\mathbb{C}) \otimes X^{\otimes n}\right)^{S_n}$

is naturally iso. to F. (see Page 7, L 15).

For the converse, let $M \in S_n$-repns.

$F = \psi(M) : \underline{\mathrm{Vect}} \to \underline{\mathrm{Vect}}$ as above. We compute

its linearization:

$L_F(X_1, \dots, X_n) =$ $(1, \dots, 1)$ component in

$$F\Big(\bigoplus_{j=1}^{n} X_j\Big) = \Big(M \otimes \Big(\bigoplus_{j=1}^{n} X_j\Big)^{\otimes n}\Big)^{S_n}$$

$$= \left(M \otimes \left[\bigoplus_{w \in S_n} X_{w(1)} \otimes \dots \otimes X_{w(n)}\right]\right)^{S_n}$$

$$\Rightarrow \varphi(\psi(M)) = L_F^{(n)}(\mathbb{C})$$

$$= \Big(M \otimes \Big[\bigoplus_{w \in S_n} \underset{\substack{w(1)\\ \text{copy}}}{\mathbb{C}} \otimes \dots \otimes \underset{\substack{w(n)\\ \text{copy}}}{\mathbb{C}}\Big] \text{ in } \underbrace{\mathbb{C} \oplus \dots \oplus \mathbb{C}}_{n\text{-terms}}\Big)^{S_n}$$

$$= \left(M \otimes \mathbb{C} S_n\right)^{S_n}$$

$\cong M$ (easy exercise: $\xi \in (M \otimes \mathbb{C} S_n)^{S_n}$ is completely det. by δ_e-term: $\xi = \sum_{w \in S_n} \underbrace{\xi_w}_{\text{in } M} \otimes \delta_w$.) $\square$

§2. Trace function. – For $F \in \mathrm{Poly}_n$, d/k and $m \in \mathbb{Z}_{\geq 1}$,

we have $\quad \mathrm{Tr}_{(m)}(F) \in \mathbb{C}[x_1, \ldots, x_m]^{S_m}_{\deg = n}$, defined by

$$\mathrm{Tr}(F) = \text{Trace of } F(\text{diagonal}(x_1, \ldots, x_m)) \text{ acting on } F(\mathbb{C}^m)$$

e.g. $F = T^2 : \underline{\mathrm{Vect}} \to \underline{\mathrm{Vect}}$ (hgs. degree 2).

If $\{v_1, \ldots, v_m\}$ is the standard basis of $\mathbb{C}^m$

$D(\underline{x}) : v_j \mapsto x_j v_j$. $\quad F(\mathbb{C}^m) = \mathbb{C}^m \otimes \mathbb{C}^m$ has basis

diagonal operator $\quad v_i \otimes v_j \leftarrow$ eigenvector for $D(\underline{x})$

$i, j \in \{1, \ldots, m\}$ w/ eigenvalue $x_i x_j$

So, $\displaystyle \mathrm{Tr}(T^2) = \sum_{i,j \in \{1,\ldots,m\}} x_i x_j = (x_1 + \cdots + x_m)^2$

More generally, $\mathrm{Tr}(T^n) = (x_1 + \cdots + x_m)^n$

Easy exercise: $\quad \mathrm{Tr}(\mathrm{Sym}^n) = h_n(x_1, \ldots, x_m)$

complete symm fn.

$\mathrm{Tr}(\Lambda^n) = e_n(x_1, \ldots, x_m)$

elementary symm. fn.

[Note $\Lambda^n(V) = 0$ iff $\dim V < n$ $\longleftrightarrow$ $e_n(x_1, \ldots, x_m) = 0$ iff $m < n$.]

<u>Lemma.</u> $F \in \mathrm{Poly}_n$; $m \in \mathbb{Z}_{\geq 1}$; $x_1, \ldots, x_m$ variables.

$D(\underline{x}) = \text{diagonal}\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix} \circlearrowright \mathbb{C}^m$

Then $\underset{F(\mathbb{C}^m)}{\mathrm{Trace}}\left(F(D(\underline{x}))\right)$ is S_m - invariant (i.e. symmetric in $x_1, \ldots, x_m$).

<u>Proof.</u> $\underset{\text{permutation action}}{S_m \circlearrowright \mathbb{C}^m} \rightsquigarrow S_m \circlearrowright F(\mathbb{C}^m)$

$w \in S_m \rightsquigarrow F(w) \in GL(F(\mathbb{C}^m))$.

Note : $F(D(w\underline{x})) = F(w)\, F(D(\underline{x}))\, F(w^{-1})$

$\Rightarrow \mathrm{Tr}(F(D(w\underline{x}))) = \mathrm{Tr}(F(D(\underline{x})))$ i.e., $\mathrm{Tr}(F)$ is symmetric in $x_1, \ldots, x_m$ variables. $\square$

§3. <u>Prop.</u> (i) $\mathrm{Tr}(F \otimes G) = \mathrm{Tr}(F)\, \mathrm{Tr}(G)$

(for $F \in \mathrm{Poly}_n$; $G \in \mathrm{Poly}_m$, $F \otimes G \in \mathrm{Poly}_{n+m}$
$(F \otimes G)(X) = F(X) \otimes G(X)$.)

(ii) $\mathrm{Tr}(F_1 \oplus F_2) = \mathrm{Tr}(F_1) + \mathrm{Tr}(F_2)$

[same proof as for group characters.]

§4. <u>Tensor vs. induction product.</u>

<u>Theorem.</u> -

$$\begin{tikzcd} \mathrm{Poly}_n \times \mathrm{Poly}_m \arrow[r, "\otimes"] \arrow[d, "{(\varphi, \varphi)}"'] & \mathrm{Poly}_{n+m} \arrow[d, "\varphi"] \\ S_n\text{-repns} \times S_m\text{-repns} \arrow[r, "*"'] & S_{n+m}\text{-repns} \end{tikzcd}$$

commutes, where

$V * W := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V \otimes W$ (induction product)

Proof.- In light of Thm §1 above, it suffices to show that $\forall\ V \in Rep(S_n)$, $W \in Rep(S_m)$, the following two functors are naturally iso.

$$X \longmapsto (V \otimes T^n X)^{S_n} \otimes (W \otimes T^m X)^{S_m} \quad : \quad \psi(V) \otimes \psi(W)$$

[ψ from Thm §1]

$$X \longmapsto \cancel{V \otimes W}\ (V * W \otimes T^{n+m} X)^{S_{n+m}}$$

(i) [For $G \circlearrowright A$, $H \circlearrowright B$; $A^G \otimes B^H \cong (A \otimes B)^{G \times H}$ - check]

So, $(V \otimes T^n X)^{S_n} \otimes (W \otimes T^m X)^{S_m}$

$$\cong (V \otimes T^n X \otimes W \otimes T^m X)^{S_n \times S_m}$$

$$\cong (V \otimes W \otimes T^{n+m} X)^{S_n \times S_m}$$

(ii) [$H < G$, $H \circlearrowright A$, $G \circlearrowright B$: $(\mathrm{Ind}_H^G (A) \otimes B)^G \cong (A \otimes \mathrm{Res}_H^G (B))^H$]

Hence $\left(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (V \otimes W) \otimes T^{n+m} X\right)^{S_{n+m}}$

$$\cong (V \otimes W \otimes T^{n+m} X)^{S_n \times S_m} \qquad \square$$

§5. <u>Corollary</u>. - Let $\{V_\lambda\}_{\lambda \vdash n} = \mathrm{Irred}_{f.d.}(S_n)$ as in Frobenius' labelling. (so $Ch(\chi_{V_\lambda}) = s_\lambda(x_1, \dots, x_m)$ Schur poly.)

Let $\{\mathbb{S}^\lambda : \underline{Vect} \to \underline{Vect}\}_{\lambda \vdash n}$ be corresponding (under Thm §1 iso.) hgs, degree n, poly. functors - <u>called Schur functors</u>. Then

$$\mathrm{Tr}(\mathbb{S}^\lambda) = s_\lambda(x_1, \dots, x_m).$$

<u>Remark</u>. - Let $K(\mathrm{Poly}_n)$ = free ab. gp. on iso. classes of hgs, degree n poly. functors.

$$= \bigoplus_{\lambda \vdash n} \mathbb{Z}\, \underbrace{[\mathbb{S}^\lambda]}_{\text{symbol for Schur functor}}$$

$$\begin{tikzcd} K(\mathrm{Poly}_n) \arrow[rr, shift left, "\varphi"] \arrow[dr, "\mathrm{Trace}"'] & & R(S_n) = \bigoplus_{\lambda \vdash n} \mathbb{Z}\, \chi_{V_\lambda} \arrow[ll, shift left, "\psi"] \arrow[dl, "Ch"] \\ & \Lambda_n = \mathbb{Z}[x_1, \dots, x_m]^{S_m}_{deg = n} & \end{tikzcd}$$

Statement of Cor. above amounts to comm. of this diagram.

<u>Proof</u>. -

We check this diagram commutes for a different $\mathbb{Z}$- basis of $R(S_n)$ - namely $U_\lambda = \mathrm{Ind}^{S_n}_{S_{\lambda_1} \times \dots \times S_{\lambda_\ell}}(\text{Trivial})$. $Ch(\chi_{U_\lambda}) = h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$.

By Thm §4, $\psi(U_\lambda) \cong \psi(\text{Trivial repn of } S_{\lambda_1}) \otimes \dots \otimes \psi(\text{Triv. of } S_{\lambda_\ell})$

Ex. $\psi(\text{Triv. of } S_\ell) = \mathrm{Sym}^\ell : \underline{Vect} \to \underline{Vect}$

and $\mathrm{Trace}(\mathrm{Sym}^\ell) = h_\ell$ (see page 3 above). □